

THE TWO DIMENSIONAL DISTRIBUTION OF VALUES OF $\zeta(1+it)$

YOUNESS LAMZOURI

ABSTRACT. We prove several results on the distribution function of $\zeta(1+it)$ in the complex plane, that is the joint distribution function of $\arg \zeta(1+it)$ and $|\zeta(1+it)|$. Similar results are also given for $L(1, \chi)$ (as χ varies over non-principal characters modulo a large prime q).

CONTENTS

Introduction	1
1. Detailed statement of results	4
2. Approximations of $\zeta(1+it)$	7
3. Estimates of sums of divisor functions	10
4. Moments of $\zeta(1+it)$	15
5. Large values of $\zeta(1+it)$ in every direction	18
6. Random Euler products and their distribution	24
7. Fourier analysis on the n -dimensional torus	28
8. The normal distribution of $\arg \zeta(1+it)$	32
9. Analogous results for $L(1, \chi)$	35
References	40

INTRODUCTION

The values of the Riemann zeta function and L -functions at the edge of the critical strip $\operatorname{Re}(s) = 1$, have important arithmetical consequences. The first one being the fact that $\zeta(1+it) \neq 0$ implies the prime number theorem, proved by Hadamard and de La Vallée Poussin in 1896, that

$$\pi(x) \sim \frac{x}{\log x}, \quad \text{as } x \rightarrow \infty.$$

The second one is the class number formula, proved by Dirichlet in 1839, which relates the class number of a quadratic extension of \mathbb{Q} to the value of $L(1, \chi_d)$ where d is the discriminant of the field extension.

The distribution of these values have been extensively studied over the last decades. One can quote the work of Granville-Soundararajan [10] in the case of $|\zeta(1+it)|$; Elliott ([7] and [8]), Montgomery-Vaughan [19] and Granville-Soundararajan [11] in the case of Dirichlet L -functions of quadratic characters $L(1, \chi_d)$; Duke [6] in the case of Artin L -functions, and the work of Cogdell-Michel [4], Habsieger-Royer [12], Lau-Wu [14], Liu-Royer-Wu

[17], Royer ([21] and [22]), and Royer-Wu ([23] and [24]) in the case of symmetric power L -functions of GL_2 -automorphic forms.

We know that the Riemann zeta function $\zeta(s)$ has a conditionally convergent Euler product on $\text{Re}(s) = 1$

$$(1) \quad \zeta(1+it) = \lim_{y \rightarrow \infty} \prod_{p \leq y} \left(1 - \frac{1}{p^{1+it}}\right)^{-1}, \text{ if } t \gg 1.$$

In 1928, assuming the Riemann Hypothesis, Littlewood ([15] and [16]) showed that one can truncate this product at $p \leq \log^2 t$ to obtain a good approximation for $\zeta(1+it)$, deducing that $|\zeta(1+it)| \leq (2e^\gamma + o(1)) \log_2 t$. (Throughout \log_j denotes the j -th iterated logarithm, so that $\log_1 n = \log n$ and $\log_j n = \log(\log_{j-1} n)$ for each $j \geq 2$). This shows that under the Riemann Hypothesis the sum $\sum_{p \geq y} 1/p^{1+it}$ is small for $y \geq \log^2 t$. Moreover using Dirichlet's Theorem on diophantine approximation it is possible to make the sum $\sum_{p \leq \log t} 1/p^{1+it}$ large, by choosing t such that $p^{it} \approx 1$, for all the primes $p \leq \log t$. This enabled Littlewood ([15] and [16]) to show the existence of arbitrarily large t for which $|\zeta(1+it)| \geq (e^\gamma + o(1)) \log_2 t$. Furthermore it is widely believed that the sum $\sum_{\log t \leq p \leq \log^2 t} 1/p^{1+it}$ is small so that the truncated product up to $\log t$ still serves as a good approximation for $\zeta(1+it)$:

Conjecture 1. *As $t \rightarrow \infty$, we have*

$$\zeta(1+it) \sim \prod_{p \leq \log t} \left(1 - \frac{1}{p^{1+it}}\right)^{-1}.$$

One consequence of this conjecture is that $\max_{|t| \leq T} |\zeta(1+it)| \sim e^\gamma \log_2 T$. In 2003, Granville and Soundararajan [10] evaluate the frequency with which such extreme values are attained, giving strong evidence for the truth of Conjecture 1. More precisely if

$$\Phi_T(\tau) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : |\zeta(1+it)| > e^\gamma \tau\},$$

then uniformly in the range $1 \ll \tau \leq \log_2 T - \log_3 T$, they proved that

$$(2) \quad \Phi_T(\tau) = \exp\left(-\frac{2e^{\tau-C-1}}{\tau} \left(1 + O\left(\frac{1}{\tau^{\frac{1}{2}}}\right)\right)\right),$$

where

$$(3) \quad C = \int_0^2 \log I_0(t) \frac{dt}{t^2} + \int_2^\infty (\log I_0(t) - t) \frac{dt}{t^2},$$

is a positive constant and $I_0(t) := \sum_{n=0}^\infty (t/2)^{2n}/n!^2$ is the modified Bessel function of order 0.

The aim of this paper is to investigate the tail of the joint distribution function of $|\zeta(1+it)|$ and $\arg \zeta(1+it)$ (where the latter is defined by continuous variation of the argument along the straight lines joining 2 , $2+it$ and $1+it$ starting with the value 0):

$$\Phi_T(\tau, \theta) := \frac{1}{T} \text{meas}\{t \in [T, 2T] : |\zeta(1+it)| > e^\gamma \tau, |\arg \zeta(1+it)| > \theta\},$$

for τ large and $\theta > 0$ bounded. In the same range $1 \ll \tau \leq \log_2 T - \log_3 T$ as in (2), we show (in Theorem 1) that for any fixed $\theta > 0$

$$(4) \quad \Phi_T(\tau, \theta) = \exp\left(-e^{\tau(1+o_\theta(1))}\right),$$

so the proportion does not decay too fast. We can be more precise showing (see Theorem 5 below), in the smaller range $1 \ll \tau \leq (\log_2 T)/2 - 2\log_3 T$, and $(\log \tau)\sqrt{\frac{\log_2 \tau}{\tau}} < \theta \ll 1$, that

$$(5) \quad \Phi_T(\tau, \theta) = \exp\left(-\frac{e^{\tau + \frac{\theta^2 \tau}{2 \log \tau} + O\left(\frac{\theta^2 \tau}{\log^2 \tau}\right)}}{\tau}\right).$$

We do prove the implicit upper bound in the full range $1 \ll \tau \leq \log T - \log 10$ unconditionally, and that the lower bound holds in this range assuming the Lang-Waldshmidt conjecture for linear forms in logarithms (Conjecture 2 below). As a consequence of our result we deduce that almost all values of $\zeta(1+it)$ with large norm are concentrated near the positive real axis:

Corollary 1. *As $\tau, T \rightarrow \infty$ with $\tau \leq \log_2 T - \log_3 T$, almost all values $t \in [T, 2T]$, with $|\zeta(1+it)| > e^\gamma \tau$, satisfy $|\arg \zeta(1+it)| \leq (\log \tau)\sqrt{\log_2 \tau / \tau}$. Moreover the set of exceptions has measure $\leq \exp(-\exp(\tau + (\log \tau \log_2 \tau)/2))$.*

Also from the estimate (5), one can deduce that the larger the arguments, the more it becomes rare to find values with large norm. More precisely we have

Corollary 2. *Let τ, θ_1 and θ_2 , be in the range of validity of Theorem 5. If τ is large and $\theta_1 > \theta_2(1 + c_5/\log \tau)$, where c_5 is a suitably large constant, then*

$$\Phi_T(\tau, \theta_1) = o(\Phi_T(\tau, \theta_2)), \text{ as } \tau, T \rightarrow \infty.$$

Let $\tau \leq \log_2 T$ be a large real number. Another interesting question is to understand the behavior of the argument of $\zeta(1+it)$ for t with $|\zeta(1+it)| \approx e^\gamma \tau$. The appearance of the factor $(\theta^2/2)\tau/\log \tau$ in (5), may suggests a normal behavior in the argument θ . Indeed we evaluate the characteristic function of $\arg \zeta(1+it)$ with an appropriate weight, and show that these arguments should be distributed according to a normal law of mean 0 and variance $\log(\tau - 1 - C)/2e^{\tau-1-C}$ (see Theorem 6 below).

We will introduce a random model for the values $\zeta(1+it)$: Let $\{X(p)\}_{p \text{ prime}}$ be a set of independent random variables, uniformly distributed on the unit circle \mathbb{U} , and define the “random Euler product”

$$L(1, X) = \lim_{y \rightarrow \infty} \prod_{p \leq y} \left(1 - \frac{X(p)}{p}\right)^{-1}, \text{ (these products converge with probability 1).}$$

Our strategy is to compare the distribution of the values of $\zeta(1+it)$ with the distribution of $L(1, X)$. For example we show in Theorem 2 below, that large complex moments of $\zeta(1+it)$ and $L(1, X)$ are roughly equal (Granville and Soundararajan (unpublished) proved an analogous result for $L(1, \chi)$, see Theorem B in section 9). Therefore we study this probabilistic model closely (Theorem 3) and deduce results on the distribution of $\zeta(1+it)$ (Theorem 5).

The results proved here carry over to $L(1, \chi)$ (where χ varies over non-principal characters modulo a large prime q) without any difficulty. We discuss these results in section 9.

Acknowledgments. I sincerely thank my advisor, Professor Andrew Granville, for suggesting this problem and for all his advice and encouragement. I would also thank Professor K. Soundararajan for valuable discussions.

1. DETAILED STATEMENT OF RESULTS

First we define

$$\zeta(1+it, y) := \prod_{p \leq y} \left(1 - \frac{1}{p^{1+it}}\right)^{-1}, \text{ and } R_y := \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1}.$$

To exhibit large values of $\zeta(1+it)$ in any given direction $\arg z = \theta$, we first approximate $\zeta(1+it)$ by short Euler products $\zeta(1+it, y)$ (which is possible for almost all $t \in [T, 2T]$ by Lemma 2.4 below in the range $1 \ll y \leq \log T$), then we try to find many values $t \in [T, 2T]$ for which

$$\zeta(1+it, y) \approx e^{i\theta} R_y.$$

To do so we use a biased method of moments, which we describe below. The first step is to note that the following inequality

$$(1.1) \quad |\zeta(1+it, y) + e^{i\theta} R_y| \geq (2-\epsilon) R_y,$$

implies

$$\zeta(1+it, y) = e^{i\theta} R_y (1 + O(\sqrt{\epsilon})).$$

This follows from the fact that $|\zeta(1+it, y)| \leq |e^{i\theta} R_y|$, and noting that for a complex number $|z| \leq 1$, with $|z+1| \geq 2-\epsilon$, one can easily show that $z = 1 + O(\sqrt{\epsilon})$. To prove

(1.1) we can try to have a good lower bound for the moments

$$(1.2) \quad \begin{aligned} & \frac{1}{T} \int_T^{2T} |\zeta(1+it, y) + e^{i\theta} R_y|^{2k} dt \\ &= \sum_{0 \leq l, m \leq k} \binom{k}{l} \binom{k}{m} R_y^{2k-l-m} e^{i\theta(m-l)} \frac{1}{T} \int_T^{2T} \zeta(1+it, y)^l \zeta(1-it, y)^m dt. \end{aligned}$$

In general, we can estimate these moments if the central terms $m = l$ constitute the main term, (since for most cases it's difficult to handle the non-central ones). However this is not the case here. In fact if $y \leq (\log T)^2$, and $m, l \leq \log T / (25 \log_2 T \log_3 T)$, then by Theorem 4.1 below, we have

$$(1.3) \quad \frac{1}{T} \int_T^{2T} \zeta(1+it, y)^l \zeta(1-it, y)^m dt = \sum_{n \in S(y)} \frac{d_l(n) d_m(n)}{n^2} + o(1),$$

and so by Proposition 3.2 below one can see that some non-central terms have the same order as the central ones. Therefore it seems difficult to estimate (1.2), because of the oscillation of $e^{i\theta(m-l)}$. To handle this, we slightly modify the moments. Indeed, instead of working with $\zeta(1+it, y)$, we search for some completely multiplicative function $f(n)$ with values on the unit circle \mathbb{U} , such that

$$\prod_{p \leq y} \left(1 - \frac{f(p)}{p^{1+it}}\right)^{-1} = R_y(1 + O(\epsilon)) \iff \zeta(1+it) = e^{i\theta} R_y(1 + O(\epsilon)).$$

In this case the non-central terms in

$$(1.4) \quad \frac{1}{T} \int_T^{2T} \left| \prod_{p \leq y} \left(1 - \frac{f(p)}{p^{1+it}}\right)^{-1} + R_y \right|^{2k} dt,$$

will be positive (by Theorem 4.1), and the central ones will give the lower bound we search for. In fact it turns out that the function we need, satisfies $f(p) = e^{-i\psi}$ for all $p \leq y$, where $\psi = \theta / \log_2 y$.

Using this method we can prove the existence of large values of $\zeta(1+it)$ in each given direction $\arg z = \theta$. Indeed we prove

Theorem 1. *Let T be large, and fix $\theta \in (-\pi, \pi]$. If $1 \ll y \leq \log T / \log_2 T$ is a real number, let $M(\theta, y)$ be the measure of values $t \in [T, 2T]$, for which*

$$\zeta(1+it) = e^{i\theta} \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1} \left(1 + O\left(\frac{1}{\log_2 y}\right)\right).$$

Then there exist two positive constants c_1, c_2 (depending on the constant in the O) for which

$$T \exp\left(-y^{1-c_2/(\log_2 y)^2}\right) \leq M(\theta, y) \leq T \exp\left(-y^{1-c_1/\log_2 y}\right).$$

Granville and Soundararajan (unpublished) used a different method to prove the existence of large values (and small values) of $L(1, \chi)$ in every direction (see Theorem A of section 9). However they only got a lower bound for the measure, and their bound is less strong than what we obtain in Theorem 1.

Let z be a complex number. We define the “ z th divisor function” $d_z(n)$, to be the multiplicative function such that $d_z(p^a) = \Gamma(z + a)/\Gamma(z)a!$, for any prime p and any integer $a \geq 0$. Then $d_z(n)$ is the coefficient of the Dirichlet series $\zeta(s)^z$ for $\text{Re}(s) > 1$. Therefore for the random variables $\{X(p)\}_{p \text{ prime}}$ we have (with probability 1) that

$$L(1, X)^z = \sum_{n=1}^{\infty} \frac{d_z(n)X(n)}{n},$$

where $X(n) = \prod_{i=1}^k X(p_i)^{a_i}$, if $n = \prod_{i=1}^k p_i^{a_i}$. If Y is a random variable on a probability space (Ω, μ) we define its expectation by $\mathbb{E}(Y) = \int_{\Omega} Y d\mu$. Therefore $\mathbb{E}(X(n)\overline{X(m)}) = 1$ if $n = m$ and vanishes otherwise. Thus for any complex numbers z_1 and z_2 , we have

$$\mathbb{E}\left(L(1, X)^{z_1} \overline{L(1, X)^{z_2}}\right) = \sum_{n=1}^{\infty} \frac{d_{z_1}(n)d_{z_2}(n)}{n^2}.$$

The idea of using a probabilistic model appears previously in the work of Montgomery-Vaughan [19], Granville-Soundararajan [11], and Cogdell-Michel [4]. Indeed in each of these cases an adequate probabilistic model was constructed to understand the distribution of appropriate L -functions. To convince ourselves that it is the right model to use, we evaluate high complex moments of $\zeta(1 + it)$ and found that

Theorem 2. *Uniformly for all complex numbers z_1, z_2 in the region $|z_1|, |z_2| \leq \log T / (50(\log_2 T)^2)$, we have*

$$\frac{1}{T} \int_T^{2T} \zeta(1 + it)^{z_1} \zeta(1 - it)^{z_2} dt = \mathbb{E}\left(L(1, X)^{z_1} \overline{L(1, X)^{z_2}}\right) + O\left(\exp\left(-\frac{\log T}{2 \log_2 T}\right)\right),$$

Using a combinatorial argument, Granville and Soundararajan [10], get a better result (in the uniformity of the range of moments) in the special case where $z_2 = z_1 = k \in \mathbb{Z}$.

This result motivated us to study the distribution of the random Euler products $L(1, X)$. For $\tau, \theta > 0$, define

$$\Phi(\tau, \theta) := \text{Prob}(|L(1, X)| > e^{\gamma} \tau, |\arg L(1, X)| > \theta).$$

A close study of this model allowed us to find a precise estimate for this distribution function. Indeed we prove the following

Theorem 3. *For $\tau > 0$ large and $(\log \tau) \sqrt{\frac{\log_2 \tau}{\tau}} < \theta \ll 1$, we have*

$$\Phi(\tau, \theta) = \exp\left(-\frac{e^{\tau + \frac{\theta^2 \tau}{2 \log \tau} + O\left(\frac{\theta^2 \tau}{\log^2 \tau}\right)}}{\tau}\right).$$

Let p_j denotes the j -th smallest prime number. To prove an analogous formula for $\Phi_T(\tau, \theta)$, we studied the behavior of the vector $V(t) := (p_1^{it}, p_2^{it}, \dots, p_N^{it})$, in the torus $\mathbb{T}^N := (\mathbb{R}/\mathbb{Z})^N$, for $t \in [T, 2T]$, as $T \rightarrow \infty$. In fact we believe that these values should be equidistributed on \mathbb{T}^N for $N = \pi(y)$ and $y \leq (1 + o(1)) \log T$. In [1], Barton, Montgomery and Vaaler an hold is constructed trigonometric polynomials in N variables, which give a sharp approximation to the characteristic function of a cartesian product of N open intervals (see Theorem 7.1). These polynomials are the analogue of Selberg polynomials in 1 variable (see [18]). Using this construction and Fourier analysis on \mathbb{T}^N , we show in Theorems 4A and 4B below, that these values are equidistributed on \mathbb{T}^N , for $y \leq \sqrt{\log T}/(\log_2 T)^2$ unconditionally, and for $y \leq (\log T)/10$ under a conjecture on linear forms in logarithms, formulated by Lang and Waldshmidt [13, Introduction to chapter X and XI, p. 212]:

Conjecture 2. *Let b_i be integers, and a_i be positive integers for which $\log a_i$ are linearly independent over \mathbb{Q} . We let $B_j = \max\{|b_j|, 1\}$, and $B = \max_{1 \leq j \leq n} B_j$. Then for any $\epsilon > 0$, there exists a positive constant $c(\epsilon)$, such that*

$$|b_1 \log a_1 + b_2 \log a_2 + \dots + b_n \log a_n| > \frac{c(\epsilon)^n B}{(B_1 \dots B_n a_1 \dots a_n)^{1+\epsilon}}.$$

More precisely we prove

Theorem 4A. *Let $2 < y$ be a real number. For each $1 \leq j \leq \pi(y)$, let $I_j \subset (0, 1)$ be an open interval of length $\delta_j > 0$. Define*

$$M(I_1, \dots, I_{\pi(y)}) = M := \text{meas} \left\{ t \in [T, 2T] : \left\{ \frac{t \log p_j}{2\pi} \right\} \in I_j, \text{ for all } 1 \leq j \leq \pi(y) \right\},$$

where $\{\cdot\}$ denotes the fractional part. Then

$$M \sim T \prod_{j \leq \pi(y)} \delta_j,$$

uniformly for $y \leq \sqrt{\log T}/(\log_2 T)^2$, and $\delta_j > (\log_2 T)^{-5/3}$.

Theorem 4B. *Assume Conjecture 2. Then with the same notations as Theorem 4A, we have*

$$M \sim T \prod_{j \leq \pi(y)} \delta_j,$$

uniformly for $y \leq (\log T)/10$, and $\delta_j > (\log T)^{-3/2}$.

Following the proof of Theorem 3 and using Theorems 4A and 4B we deduce

Theorem 5. *Let $T > 0$ be large. There exists two positive constants c_3 and c_4 for which*

$$\Phi_T(\tau, \theta) \leq \exp \left(- \frac{e^{\tau + \frac{\theta^2 \tau}{2 \log \tau} - c_3 \frac{\theta^2 \tau}{\log^2 \tau}}}{\tau} \right),$$

uniformly for $1 \ll \tau \leq \log_2 T$, and $(\log \tau) \sqrt{\frac{\log_2 \tau}{\tau}} < \theta \ll 1$. And

$$\Phi_T(\tau, \theta) \geq \exp \left(- \frac{e^{\tau + \frac{\theta^2 \tau}{2 \log \tau} + c_4 \frac{\theta^2 \tau}{\log^2 \tau}}}{\tau} \right),$$

uniformly for $(\log \tau) \sqrt{\frac{\log_2 \tau}{\tau}} < \theta \ll 1$, and $1 \ll \tau \leq (\log_2 T)/2 - 2 \log_3 T$ unconditionally, and for $1 \ll \tau \leq \log_2 T - \log 10$ if we assume Conjecture 2.

We now turn our attention to the behavior of $\arg \zeta(1+it)$ when the norm is large, that is when $|\zeta(1+it)| \approx e^\gamma \tau$ with $\tau \leq (1+o(1)) \log_2 T$. We compute the characteristic function of these arguments with a natural weight, and use the Berry-Esseen Theorem ([2], [9]) to prove the following

Theorem 6. *Let $T > 0$ be large, $1 \ll \tau \leq \log_2 T - 3 \log_3 T$ a real number, $\epsilon = \tau^{-1/5}$ and $k = e^{\tau-1-C}$, where C is defined by (3). Let*

$$\Omega_T(\tau) := \{t \in [T, 2T] : e^\gamma(\tau - \epsilon) \leq |\zeta(1+it)| \leq e^\gamma(\tau + \epsilon)\},$$

and for a real number x , let

$$\Lambda_T(\tau, x) := \left\{ t \in \Omega_T(\tau) : \frac{\arg \zeta(1+it)}{\sqrt{\frac{\log(\tau-1-C)}{2e^{\tau-1-C}}}} < x \right\}, \text{ and } \nu_{T,\tau}(x) := \frac{\int_{\Lambda_T(\tau,x)} |\zeta(1+it)|^{2k} dt}{\int_{\Omega_T(\tau)} |\zeta(1+it)|^{2k} dt}.$$

Then we have

$$\nu_{T,\tau}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy + O_x \left(\frac{1}{\sqrt{\log \tau}} \right).$$

2. APPROXIMATIONS OF $\zeta(1+it)$

2.1 Short Euler product approximation.

In this section we approximate $\zeta(1+it)$ by a short Euler product of length $y \leq \log T$, for almost all $t \in [T, 2T]$. The main idea is to show that this is possible if $\zeta(s)$ has no zeros far from the critical line, then to use a classical zero-density estimate (there are few such zeros) to see that we can almost surely avoid there zeros. The material of this section is classical, and it's essentially proved in Granville and Soundararajan [10] (see sections 2 and 3).

Lemma 2.1 ([10, Lemma 1]). *Let $y \geq 2$ and $|t| \geq y+3$ be real numbers. Let $\frac{1}{2} \leq \sigma_0 < 1$ and suppose that the rectangle $\{s : \sigma_0 < \operatorname{Re}(s) \leq 1, |\operatorname{Im}(s) - t| \leq y+2\}$ does not contain any zeros of $\zeta(s)$. Then if $\sigma_0 < \sigma \leq 2$ and $|x - t| \leq y$ we have*

$$|\log \zeta(\sigma + ix)| \ll \log |t| \log(e/(\sigma - \sigma_0)).$$

Moreover, if $\sigma_0 < \sigma \leq 1$ then

$$\log \zeta(\sigma + it) = \sum_{n=2}^y \frac{\Lambda(n)}{n^{\sigma+it} \log n} + O\left(\frac{\log |t|}{(\sigma_1 - \sigma_0)^2} y^{\sigma_1 - \sigma}\right),$$

where $\sigma_1 = \min(\sigma_0 + \frac{1}{\log y}, \frac{\sigma + \sigma_0}{2})$.

From this result, we deduce

Lemma 2.2 ([10, Lemma 2]). *Let $\frac{1}{2} < \sigma \leq 1$ be fixed, T large and $3 < y < T/2$ be a real number. We have*

$$\log \zeta(\sigma + it) = \sum_{n=2}^y \frac{\Lambda(n)}{n^{\sigma+it} \log n} + O(y^{(\frac{1}{2}-\sigma)/2} \log^3 T)$$

for all $t \in (T, 2T)$ except for a set of measure $\ll T^{5/4-\sigma/2} y (\log T)^5$.

Proof. This follows from combining the classical zero-density estimate $N(\sigma_0, T) \ll T^{3/2-\sigma_0} (\log T)^5$ (see Theorem 9.19 A of [25]) and Lemma 2.1 (taking $\sigma_0 = (1/2 + \sigma)/2$ there).

To obtain an approximation by shorter Euler products, we need a large sieve type inequality

Lemma 2.3 ([10, Lemma 3]). *Let $2 \leq y \leq z$ be real numbers. For arbitrary complex numbers $x(p)$ we have*

$$\frac{1}{T} \int_T^{2T} \left| \sum_{y \leq p \leq z} \frac{x(p)}{p^{it}} \right|^{2k} dt \ll \left(k \sum_{y \leq p \leq z} |x(p)|^2 \right)^k + T^{-\frac{2}{3}} \left(\sum_{y \leq p \leq z} |x(p)| \right)^{2k}$$

uniformly for all integers $1 \leq k \leq \log T / (3 \log z)$.

We define $\zeta(s, y) := \prod_{p \leq y} (1 - p^{-s})^{-1}$, and using the Lemmas above, we prove the following key Lemma

Lemma 2.4. *Let $T > 0$ be a large real number, and $A(t) \leq \log t$ be a slowly increasing function which tends to ∞ with t . Then, uniformly for $y \leq \log T$, we have*

$$\zeta(1+it) = \zeta(1+it, y) \left(1 + O\left(\frac{1}{A(y)}\right) \right),$$

for all $t \in [T, 2T]$ except a set of measure

$$\ll T \exp\left(-\log\left(\frac{300 \log^2 y}{A(y)^2}\right) \frac{y}{300 \log y}\right).$$

Proof. Let $z = (\log T)^{100}$, we deduce from Lemma 2.2 that

$$(2.1) \quad \zeta(1+it) = \zeta(1+it, z) \left(1 + O\left(\frac{1}{\log T}\right) \right),$$

for all $t \in [T, 2T]$ except a set of measure at most $T^{4/5}$. Applying Lemma 2.3 with $x(p) = 1/p$, we get

$$\frac{1}{T} \int_T^{2T} \left| \sum_{y \leq p \leq z} \frac{1}{p^{1+it}} \right|^{2k} dt \ll \left(k \sum_{y \leq p \leq z} \frac{1}{p^2} \right)^k + T^{-2/3} \left(\sum_{y \leq p \leq z} \frac{1}{p} \right)^{2k},$$

for any integer $1 \leq k \leq \log T / 3 \log z$. We choose $k = [y / (300 \log y)]$, which implies that

$$(2.2) \quad \frac{1}{T} \int_T^{2T} \left| \sum_{y \leq p \leq z} \frac{1}{p^{1+it}} \right|^{2k} dt \ll \left(\frac{1}{300 \log^2 y} \right)^k.$$

Let $M = \text{meas}\{t \in [T, 2T] : \left| \sum_{y \leq p \leq z} \frac{1}{p^{1+it}} \right| > \frac{1}{A(y)}\}$. From (2.2) we get

$$\frac{M}{T} \left(\frac{1}{A(y)} \right)^{2k} \ll \left(\frac{1}{300 \log^2 y} \right)^k,$$

which implies that

$$M \ll T \exp \left(-\log \left(\frac{300 \log^2 y}{A(y)^2} \right) \frac{y}{300 \log y} \right).$$

To complete the proof one may check that for all $t \in [T, 2T]$, except a set of measure M , we have

$$\begin{aligned} \zeta(1+it, z) &= \zeta(1+it, y) \exp \left(- \sum_{y \leq p \leq z} \left(\frac{1}{p^{1+it}} + O \left(\frac{1}{p^2} \right) \right) \right) \\ &= \zeta(1+it, y) \left(1 + O \left(\frac{1}{A(y)} \right) \right). \end{aligned}$$

2.2 Smooth Dirichlet series approximation of $\zeta(1+it)^z$.

To prove Theorem 2, we need the following Lemma, which corresponds to Lemma 2.3 of Granville-Soundararajan [11].

Lemma 2.5. *Let t be large, and z be any complex number with $|z| \leq \log^2 t$. Define $Z = \exp((\log t)^{10})$. Then*

$$\zeta(1+it)^z = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^{1+it}} e^{-n/Z} + O \left(\frac{1}{t} \right).$$

Proof. Since $\frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} y^s \Gamma(s) ds = e^{-1/y}$, we have

$$(2.3) \quad \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \zeta(1+it+s)^z Z^s \Gamma(s) ds = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^{1+it}} e^{-n/Z}.$$

We shift the line of integration to the contour $s = -C(x) + ix$ where $C(x) := c/(2 \log(|x| + 2))$, and $c > 0$ is chosen so that $\zeta(s)$ have no zeros in the region where we shift the contour (this is possible by the classical theorem on the zero free region of $\zeta(s)$). We encounter a pole at $s = 0$, which leaves the residue $\zeta(1+it)^z$. Applying Lemma 2.1 with $\sigma_0 = 1 - 4C(x)/3$ and $y = 2$ gives $|\log \zeta(s)| \ll 1/C(x)^2$ and so the integral along the new contour is

$$\ll \int_{-\infty}^{\infty} Z^{-C(x)} e^{O(|z|/C(x)^2)} |\Gamma(-C(x) + ix)| dx \ll \frac{1}{t},$$

by Stirling's formula. This completes the proof.

3. ESTIMATES OF SUMS OF DIVISOR FUNCTIONS

In this section we prove two results on sums of the divisor function $d_k(n)$. The advantage of our results is the uniformity on k . We begin by proving the following proposition on the estimates of such sums in short intervals, which we shall use later in the proof of Theorem 2

Proposition 3.1. *Let $T > 0$ be a large real number, and $k \leq \log T/4(\log_2 T)^2$ a positive integer. Define $Z = \exp((\log T)^{10})$ and $y = \exp(\log T/\log_2 T)$. Then*

$$\sup_{m > \sqrt{T}} \sum_{m < n < m + \frac{m}{\sqrt{T}}} \frac{d_k(n)}{n} e^{-n/Z} \leq \frac{(\log 3Z)^k}{y}.$$

Proof. We prove this by induction on k . If $k = 1$ then

$$\sup_{m > \sqrt{T}} \sum_{m < n < m + \frac{m}{\sqrt{T}}} \frac{e^{-n/Z}}{n} \leq \sup_{m > \sqrt{T}} \frac{1}{m} \left(\frac{m}{\sqrt{T}} + 1 \right) \leq \frac{2}{\sqrt{T}} \leq \frac{\log 3Z}{y}.$$

Now suppose the result true for $k - 1$. K. Norton [20] proved that

$$\log d_k(n) \leq \frac{\log n \log k}{\log \log n} \left(1 + \frac{\log \log \log n}{\log \log n} \left(1 + O\left(\frac{1}{\log \log n} \right) \right) \right),$$

uniformly for $k \leq \log n/(\log \log n)^2$, if n is large enough. Thus

$$(3.1) \quad \begin{aligned} & \sup_{\sqrt{T} < m < y\sqrt{T}} \sum_{m < n < m + \frac{m}{\sqrt{T}}} \frac{d_k(n)}{n} e^{-n/Z} \leq \sup_{\sqrt{T} < m < y\sqrt{T}} \sum_{m < n < m + y} \frac{d_k(n)}{n} \\ & \leq \sup_{\sqrt{T} < m < y\sqrt{T}} \left(y \max_{m < n < m + y} \frac{d_k(n)}{n} \right) \leq \frac{1}{y}. \end{aligned}$$

Now for $m > y\sqrt{T}$, we have

$$\sum_{m < n < m + \frac{m}{\sqrt{T}}} \frac{d_k(n)}{n} e^{-n/Z} = \sum_{m < n < m + \frac{m}{\sqrt{T}}} \frac{e^{-n/Z}}{n} \sum_{dr=n} d_{k-1}(r).$$

We divide the above sum into two parts: S_1 for $d > y$ (which implies that $r \leq \frac{2m}{y}$), and S_2 , for $d \leq y$. We have then

$$S_1 \leq \sum_{r \leq \frac{2m}{y}} \frac{d_{k-1}(r)}{r} e^{-r/Z} \sum_{\frac{m}{r} < d < \frac{m}{r} + \frac{m}{r\sqrt{T}}} \frac{1}{d} \leq \sum_{r \leq \frac{2m}{y}} \frac{d_{k-1}(r)}{r} e^{-r/Z} \left(\frac{1}{\sqrt{T}} + \frac{2}{y} \right).$$

For $j \in \mathbb{N}$, we have that $d_j(n) e^{-n/Z} \leq e^{j/Z} \sum_{a_1 \dots a_j = n} e^{-(a_1 + \dots + a_j)/Z}$, and so

$$(3.2) \quad \sum_{n=1}^{\infty} \frac{d_j(n)}{n} e^{-n/Z} \leq \left(e^{1/Z} \sum_{a=1}^{\infty} \frac{e^{-a/Z}}{a} \right)^j \leq (\log 3Z)^j.$$

This implies

$$(3.3) \quad S_1 \leq \frac{3(\log 3Z)^{k-1}}{y} \leq \frac{(\log 3Z)^k}{3y}.$$

Moreover

$$S_2 \leq \sum_{d \leq y} \frac{1}{d} \sum_{\frac{m}{d} < r < \frac{m}{d} + \frac{m}{d\sqrt{T}}} \frac{d_{k-1}(r)}{r} e^{-r/Z}.$$

Since $m > y\sqrt{T}$, and $d \leq y$, we get $m/d > \sqrt{T}$. By our induction hypothesis we deduce that

$$\sum_{\frac{m}{d} < r < \frac{m}{d} + \frac{m}{d\sqrt{T}}} \frac{d_{k-1}(r)}{r} e^{-r/Z} \leq \sup_{s > \sqrt{T}} \sum_{s < r < s + \frac{s}{\sqrt{T}}} \frac{d_{k-1}(r)}{r} e^{-r/Z} \leq \frac{(\log 3Z)^{k-1}}{y}.$$

Finally we have

$$(3.4) \quad S_2 \leq \frac{(\log 3Z)^{k-1} \log y}{y} \leq \frac{(\log 3Z)^k}{3y}.$$

Now combining (3.1), (3.3) and (3.4) gives the result.

The key ingredient of the proof of Theorem 6, is to understand the ratio

$$\sum_{n=1}^{\infty} \frac{d_{k-r}(n) d_{k+r}(n)}{n^2} \bigg/ \sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^2},$$

for large k and r . To this end we prove the following result

Proposition 3.2. *Let k be a large real number and*

$c_0 = \lim_{x \rightarrow \infty} \left(\sum_{p \leq x} 1/p - \log_2 x \right)$. Then uniformly for $|r| \leq \sqrt{k}$, we have

$$\sum_{n \geq 1} \frac{d_{k-r}(n)d_{k+r}(n)}{n^2} = \exp \left(-r^2 \frac{\log_2 k}{k} - \frac{c_0 r^2}{k} + O \left(\frac{r^2}{k\sqrt{\log k}} + \frac{r^4}{k^2} \right) \right) \sum_{n \geq 1} \frac{d_k^2(n)}{n^2}.$$

First, we remark that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - \frac{e^{i\theta}}{p} \right)^{-k} \left(1 - \frac{e^{-i\theta}}{p} \right)^{-k} d\theta \\ (3.5) \quad &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{a=0}^{\infty} \frac{d_k(p^a) e^{ia\theta}}{p^a} \sum_{b=0}^{\infty} \frac{d_k(p^b) e^{-ib\theta}}{p^b} d\theta = \sum_{a=0}^{\infty} \frac{d_k^2(p^a)}{p^{2a}}. \end{aligned}$$

Analogously we have

$$\begin{aligned} \sum_{a=0}^{\infty} \frac{d_{k-r}(p^a)d_{k+r}(p^a)}{p^{2a}} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} \left(1 - \frac{e^{i\theta}}{p} \right)^r \left(1 - \frac{e^{-i\theta}}{p} \right)^{-r} d\theta \\ &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} \left(\left(1 - \frac{e^{i\theta}}{p} \right)^r \left(1 - \frac{e^{-i\theta}}{p} \right)^{-r} + \left(1 - \frac{e^{-i\theta}}{p} \right)^r \left(1 - \frac{e^{i\theta}}{p} \right)^{-r} \right) d\theta \\ (3.6) \quad &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} \cos \left(2r \arg \left(1 - \frac{e^{i\theta}}{p} \right) \right) d\theta, \end{aligned}$$

since the series $\sum_{a=0}^{\infty} d_{k-r}(p^a)d_{k+r}(p^a)/p^{2a}$ is real. The proof will rely on these two identities. The last ingredient we need is the following Lemma

Lemma 3.3. *Let $k > 0$ be a large real number. Suppose that $p = o(k)$ as $k \rightarrow \infty$, and let $\epsilon = 4\sqrt{\frac{p}{k} \log \left(\frac{k}{p} \right)}$. Then we have*

$$\frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta \right) \left(1 + O \left(\frac{p^4}{k^4} \right) \right).$$

Proof. First we observe that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k} d\theta \\ &= \frac{1}{\pi} \int_0^{\pi} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k} d\theta. \end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{\pi} \int_{\epsilon}^{\pi/2} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2}\right)^{-k} d\theta &\leq \frac{1}{\sin \epsilon} \left(\frac{1}{\pi} \int_{\epsilon}^{\pi/2} \sin \theta \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2}\right)^{-k} d\theta \right) \\
&= \frac{1}{\sin \epsilon} \left(-\frac{p}{2\pi(k-1)} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2}\right)^{-k+1} \Big|_{\epsilon}^{\pi/2} \right) \\
&\leq \frac{p}{\pi \epsilon (k-1)} \left(1 - \frac{1}{p}\right)^{-2k+2} \exp\left(-\frac{\epsilon^2 k}{p} + O\left(\frac{\epsilon^4 k}{p}\right)\right).
\end{aligned}$$

Moreover since

$$\frac{1}{\pi} \int_{\pi/2}^{\pi} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2}\right)^{-k} d\theta \leq \frac{1}{2} \left(1 + \frac{1}{p^2}\right)^{-k},$$

then

$$E := \frac{1}{\pi} \int_{\epsilon}^{\pi} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2}\right)^{-k} d\theta \leq \frac{2p}{\pi \epsilon (k-1)} \left(1 - \frac{1}{p}\right)^{-2k+2} \exp\left(-\frac{\epsilon^2 k}{p} + O\left(\frac{\epsilon^4 k}{p}\right)\right).$$

Let $0 < \delta < \epsilon$ be a small real number, to be chosen later. We have

$$\begin{aligned}
I := \frac{1}{\pi} \int_0^{\epsilon} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2}\right)^{-k} d\theta &\geq \frac{1}{\pi} \int_0^{\delta} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2}\right)^{-k} d\theta \\
&\geq \frac{1}{\sin \delta} \left(-\frac{p}{2\pi(k-1)} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2}\right)^{-k+1} \Big|_0^{\delta} \right) \\
&\geq \frac{p}{2\pi \delta (k-1)} \left(1 - \frac{1}{p}\right)^{-2k+2} \left(1 - \exp\left(-\frac{\delta^2 k}{2p}\right)\right).
\end{aligned}$$

One may chose $\delta = \sqrt{\frac{p}{2k}}$, which implies that

$$E \leq \left(\frac{20\delta}{\epsilon} \exp\left(-\frac{\epsilon^2 k}{2p}\right)\right) I \leq \left(\frac{p^4}{k^4}\right) I,$$

completing the proof.

Proof of Proposition 3.2. If $-\pi/2 \leq \omega \leq \pi/2$, then

$$\omega = \sin \omega + O(\sin^3 \omega).$$

Also since $\cos \arg\left(1 - \frac{e^{i\theta}}{p}\right) = \frac{1 - \frac{\cos \theta}{p}}{\left|1 - \frac{e^{i\theta}}{p}\right|} > 0$, and $\sin \arg\left(1 - \frac{e^{i\theta}}{p}\right) = \frac{-\sin \theta}{p \left|1 - \frac{e^{i\theta}}{p}\right|}$, then

$$\arg\left(1 - \frac{e^{i\theta}}{p}\right) = \frac{-\sin \theta}{p \left|1 - \frac{e^{i\theta}}{p}\right|} + O\left(\frac{\sin^3 \theta}{p^3}\right).$$

This implies

$$(3.7) \quad \cos \left(2r \arg \left(1 - \frac{e^{i\theta}}{p} \right) \right) = 1 - \frac{2r^2 \sin^2 \theta}{p^2 \left| 1 - \frac{e^{i\theta}}{p} \right|^2} + O \left(\frac{(r^4 + r^2) \sin^4 \theta}{p^4} \right).$$

Now suppose that $p \leq k/\sqrt{\log k}$, in this case one has

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} \cos \left(2r \arg \left(1 - \frac{e^{i\theta}}{p} \right) \right) d\theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta \\ &- \frac{r^2}{p^2 \pi} \int_{-\pi}^{\pi} \sin^2 \theta \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k-2} d\theta + O \left(\frac{r^4 + r^2}{p^4} \int_{-\pi}^{\pi} \sin^4 \theta \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta \right). \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} \int_{-\pi}^{\pi} \sin^2 \theta \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k-2} d\theta &= \int_{-\pi}^{\pi} \sin^2 \theta \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k-1} d\theta \\ &= \frac{p}{2k} \int_{-\pi}^{\pi} \cos \theta \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k} d\theta - \frac{p}{2k} \sin \theta \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k} \Big|_{-\pi}^{\pi} \\ &= \frac{p}{2k} \int_{-\pi}^{\pi} \cos \theta \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k} d\theta. \end{aligned}$$

Further by Lemma 3.3, taking $\epsilon = 4\sqrt{\frac{p}{k} \log \left(\frac{k}{p} \right)}$ there, we get

$$\begin{aligned} \int_{-\pi}^{\pi} \cos \theta \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k} d\theta &= \left(1 + O \left(\frac{p^4}{k^4} \right) \right) \int_{-\epsilon}^{\epsilon} \cos \theta \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k} d\theta \\ &= \int_{-\epsilon}^{\epsilon} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k} d\theta \left(1 + O \left(\epsilon^2 + \frac{p^4}{k^4} \right) \right) \\ &= \int_{-\pi}^{\pi} \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k} d\theta \left(1 + O \left(\frac{p}{k} \log \left(\frac{k}{p} \right) \right) \right). \end{aligned}$$

So we deduce that

$$\int_{-\pi}^{\pi} \sin^2 \theta \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k-2} d\theta = \frac{p}{2k} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta \left(1 + O \left(\frac{p}{k} \log \left(\frac{k}{p} \right) \right) \right).$$

Moreover following the same ideas, and integrating by parts, we get

$$\begin{aligned} \int_{-\pi}^{\pi} \sin^4 \theta \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta &= \frac{3p}{2(k+1)} \int_{-\pi}^{\pi} \sin^2 \theta \cos \theta \left(1 - \frac{2 \cos \theta}{p} + \frac{1}{p^2} \right)^{-k+1} d\theta \\ &\ll \frac{p^2}{k^2} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta. \end{aligned}$$

Thus by (3.7), the RHS of (3.6) equals

$$(3.8) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta \left(1 - \frac{r^2}{pk} + O\left(\frac{r^2}{k^2} \log\left(\frac{k}{p} \right) + \frac{r^4 + r^2}{p^2 k^2} \right) \right).$$

Now for the case $p \geq k/\sqrt{\log k}$, by (3.7) we use the following estimate for the RHS of (3.6)

$$(3.9) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| 1 - \frac{e^{i\theta}}{p} \right|^{-2k} d\theta \left(1 + O\left(\frac{r^2}{p^2} \right) \right).$$

Finally upon using (3.5), (3.6) and the estimates (3.8) and (3.9) for the appropriate cases, we deduce that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{d_{k-r}(n)d_{k+r}(n)}{n^2} &= \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^2} \prod_{p \geq k/\sqrt{\log k}} \exp\left(O\left(\frac{r^2}{p^2} \right) \right) \\ &\quad \prod_{p \leq k/\sqrt{\log k}} \exp\left(-\frac{r^2}{pk} + O\left(\frac{r^2}{k^2} \log\left(\frac{k}{p} \right) + \frac{r^4 + r^2}{p^2 k^2} \right) \right) \\ &= \exp\left(-r^2 \frac{\log_2 k}{k} - \frac{c_0 r^2}{k} + O\left(\frac{r^2}{k\sqrt{\log k}} + \frac{r^4}{k^2} \right) \right) \sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^2}, \end{aligned}$$

completing the proof.

4. MOMENTS OF $\zeta(1 + it)$

In this section we prove Theorem 2 together with a result on moments of short Euler products. We begin by the proof of Theorem 2

Proof of Theorem 2. First by Lemma 2.5, for t large enough we have

$$\zeta(1 + it)^z = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^{1+it}} e^{-n/Z} + O\left(\frac{1}{t} \right),$$

where z is any complex number such that $|z| \leq (\log t)^2$, and $Z = \exp((\log t)^{10})$.

Now let $x = \max\{|z_1|, |z_2|\}$, and $k = [x] + 1$. Therefore we have

$$(4.1) \quad \begin{aligned} &\frac{1}{T} \int_T^{2T} \zeta(1 + it)^{z_1} \zeta(1 - it)^{z_2} dt \\ &= \sum_{m, n \geq 1} \frac{d_{z_1}(n) d_{z_2}(m) e^{-(m+n)/Z}}{mn} \frac{1}{T} \int_T^{2T} \left(\frac{m}{n} \right)^{it} dt + E_1, \end{aligned}$$

where

$$(4.2) \quad E_1 \ll \frac{1}{T} \sum_{n=1}^{\infty} \frac{d_k(n)}{n} e^{-n/Z} \ll \frac{(\log 3Z)^k}{T},$$

by (3.2). The series in the RHS of (4.1) includes diagonal terms $m = n$ which contribute as the main term, and off-diagonal terms $m \neq n$ which contribute as an error term, as we shall prove later. The diagonal terms contribution equals

$$\sum_{n \geq 1} \frac{d_{z_1}(n)d_{z_2}(n)e^{-2n/Z}}{n^2} = \sum_{n \geq 1} \frac{d_{z_1}(n)d_{z_2}(n)}{n^2} + E_2,$$

where

$$(4.3) \quad E_2 \ll \frac{1}{\sqrt{Z}} \sum_{n \geq 1} \frac{d_k(n)^2}{n^{3/2}} \leq \frac{1}{\sqrt{Z}} \sum_{n \geq 1} \frac{d_{k^2}(n)}{n^{3/2}} = \frac{\zeta(3/2)^{k^2}}{\sqrt{Z}},$$

knowing that $1 - e^{-t} \leq 2\sqrt{t}$ for all $t > 0$.

For the off-diagonal terms, we divide the sum into four parts:

a) $m, n \leq \sqrt{T}$, b) $m \geq n + n/\sqrt{T}$, c) $n \geq m + m/\sqrt{T}$, and d) whatever remains. For the three first cases we use the following inequality

$$(4.4) \quad \frac{1}{T} \int_T^{2T} \left(\frac{m}{n}\right)^{it} dt \ll \frac{1}{T|\log(m/n)|} \leq T^{-1/2},$$

which holds since $|\log(1-c)| = -\log(1-c) > c$ for any real number $0 < c < 1$. Thus by (3.2) the contribution of such terms is

$$(4.5) \quad E_3 \ll \frac{1}{T^{1/2}} \left(\sum_{n=1}^{\infty} \frac{d_k(n)}{n} e^{-n/Z} \right)^2 \leq \frac{(\log 3Z)^{2k}}{T^{1/2}}.$$

It remains then, to bound the contribution from the last part E_4 . we have

$$|E_4| \leq \sum_{\substack{n+n/\sqrt{T} > m > n > \sqrt{T} \\ \text{or } m+m/\sqrt{T} > n > m > \sqrt{T}}} \frac{d_{|z_1|}(n)d_{|z_2|}(m)e^{-(n+m)/Z}}{mn}.$$

Let $y = \exp(\log T / \log_2 T)$. By Proposition 3.1 and (3.2), we get

$$\begin{aligned} |E_4| &\leq 2 \sum_{m > \sqrt{T}} \frac{d_k(m)}{m} e^{-m/Z} \sum_{m < n < m+m/\sqrt{T}} \frac{d_k(n)}{n} e^{-n/Z} \\ &\leq 2 \left(\sum_{m > \sqrt{T}} \frac{d_k(m)}{m} e^{-m/Z} \right) \left(\sup_{r > \sqrt{T}} \sum_{r < n < r+r/\sqrt{T}} \frac{d_k(n)}{n} e^{-n/Z} \right) \\ &\leq 2(\log 3Z)^k \frac{(\log 3Z)^k}{y} = \frac{2(\log 3Z)^{2k}}{y}. \end{aligned}$$

This gives along with (4.2), (4.3) and (4.5), the following bound

$$E_1 + E_2 + E_3 + E_4 \ll (2 \log 3Z)^{2k} / y \leq y^{-1/2},$$

which achieves the proof.

Now to prove Theorem 1, we need a similar result to Theorem 2, but for general short Euler products of degree 1. Indeed we have

Theorem 4.1. *Let $T > 0$ be large, $y \leq (\log T)^2$ a real number, and f a completely multiplicative function with values on the unit circle \mathbb{U} ($|f(n)| = 1$ for all $n \in \mathbb{N}$). Let*

$$L_f(s, y) := \prod_{p \leq y} \left(1 - \frac{f(p)}{p^s}\right)^{-1} = \sum_{n \in S(y)} \frac{f(n)}{n^s},$$

where $S(y) = \{n \in \mathbb{N} : p|n \implies p \leq y\}$. If z_1, z_2 are complex numbers verifying $|z_1|, |z_2| \leq \log T / (25 \log_2 T \log_3 T)$, then

$$\frac{1}{T} \int_T^{2T} L_f(1+it, y)^{z_1} \overline{L_f(1+it, y)^{z_2}} dt = \sum_{n \in S(y)} \frac{d_{z_1}(n) d_{z_2}(n)}{n^2} + O\left(\exp\left(-\frac{\log T}{4 \log_2 T}\right)\right).$$

Proof. Let $x = \max\{|z_1|, |z_2|\}$, and $k = [x] + 1$. Then

$$\frac{1}{T} \int_T^{2T} L_f(1+it, y)^{z_1} \overline{L_f(1+it, y)^{z_2}} dt = \sum_{m, n \in S(y)} \frac{d_{z_1}(n) d_{z_2}(m) f(n) \overline{f(m)}}{mn} \frac{1}{T} \int_T^{2T} \left(\frac{m}{n}\right)^{it} dt.$$

In this series, the diagonal terms $m = n$ contribute

$$\sum_{n \in S(y)} \frac{d_{z_1}(n) d_{z_2}(n) f(n) \overline{f(n)}}{n^2} = \sum_{n \in S(y)} \frac{d_{z_1}(n) d_{z_2}(n)}{n^2}.$$

Furthermore we divide the off-diagonal terms into two parts:

a) If $m, n \leq T^{3/4}$, and b) if $m > T^{3/4}$, or $n > T^{3/4}$. Now for the first case we have

$$\frac{1}{T} \int_T^{2T} \left(\frac{m}{n}\right)^{it} dt \ll \frac{1}{T |\log(m/n)|} \leq T^{-1/4},$$

since $|\log(1-c)| = -\log(1-c) > c$ for any real number $0 < c < 1$. Thus the contribution of such terms is bounded by

$$(4.6) \quad \frac{1}{T^{1/4}} \sum_{m, n \in S(y)} \frac{d_{|z_1|}(n) d_{|z_2|}(m)}{mn} \leq \frac{1}{T^{1/4}} \left(\sum_{n \in S(y)} \frac{d_k(n)}{n} \right)^2 \leq \frac{(3 \log y)^{2k}}{T^{1/4}}.$$

Moreover the contribution from the second part is bounded by

$$(4.7) \quad 2 \sum_{n, m \in S(y), n > T^{3/4}} \frac{d_k(n) d_k(m)}{mn} \leq 2 \left(T^{3/4}\right)^{-\alpha} \sum_{m \in S(y)} \frac{d_k(m)}{m} \sum_{n \in S(y)} \frac{d_k(n)}{n^{1-\alpha}},$$

for all $\alpha > 0$. We choose $\alpha = 1/\log_2 T$. Therefore

$$(4.8) \quad \sum_{n \in S(y)} \frac{d_k(n)}{n^{1-\alpha}} = \prod_{p \leq y} \left(1 - \frac{1}{p^{1-\alpha}}\right)^{-k} = \exp\left(k \sum_{p \leq y} \frac{1}{p^{1-\alpha}} + O(k)\right) \leq \exp(9k \log_2 y).$$

Finally by (4.6), (4.7) and (4.8) the contribution of the off-diagonal part is at most

$$\exp(-3 \log T / 4 \log_2 T + 10k \log_2 y) \leq \exp(-\log T / 4 \log_2 T),$$

proving the Theorem.

5. LARGE VALUES OF $\zeta(1+it)$ IN EVERY DIRECTION

In this section we prove Theorem 1. For $s \in \mathbb{C}$, define

$$L_\psi(s, y) := \prod_{p \leq y} \left(1 - \frac{e^{-i\psi}}{p^s}\right)^{-1}.$$

We have

Lemma 5.1. *For $\theta \in (-\pi, \pi]$, and $y > 0$ large enough, let $\psi = \theta/\log_2 y$. Then for all $t \in \mathbb{R}$, we have*

$$L_\psi(1+it, y) = R_y \left(1 + O\left(\frac{1}{\log_2 y}\right)\right) \iff \zeta(1+it, y) = e^{i\theta} R_y \left(1 + O\left(\frac{1}{\log_2 y}\right)\right).$$

Proof. First we have

$$\begin{aligned} \frac{1}{R_y} \prod_{p \leq y} \left(1 - \frac{e^{i\psi}}{p}\right)^{-1} &= \prod_{p \leq y} \left(1 - \frac{e^{i\psi} - 1}{p-1}\right)^{-1} = \exp \left(- \sum_{p \leq y} \log \left(1 - \frac{e^{i\psi} - 1}{p-1}\right) \right) \\ (5.1) \quad &= \exp \left(\sum_{p \leq y} \frac{i\psi}{p-1} + O(\psi^2 \log_2 y) \right) = e^{i\theta} \left(1 + O\left(\frac{1}{\log_2 y}\right)\right). \end{aligned}$$

Now using that $(e^{i\psi})^m = 1 + O(m\psi)$ for all $m \in \mathbb{N}$, we deduce that

$$\begin{aligned} \log \left(\prod_{p \leq y} \left(1 - \frac{e^{-i\psi}}{p^{1+it}}\right)^{-1} \right) - \log R_y &= \sum_{p \leq y} \sum_{m \geq 1} \frac{(p^{-it} e^{-i\psi})^m - 1}{p^m m} \\ &= \sum_{p \leq y} \sum_{m \geq 1} \frac{(p^{-it})^m e^{-i\psi} - 1}{p^m m} + O \left(\psi \sum_{p \leq y} \sum_{m \geq 2} \frac{1}{p^m} \right) \\ (5.2) \quad &= e^{-i\psi} \sum_{p \leq y} \sum_{m \geq 1} \frac{(p^{-it})^m - e^{i\psi}}{p^m m} + O \left(\frac{1}{\log_2 y} \right). \end{aligned}$$

Moreover by (5.1) we get

$$\begin{aligned} \log \zeta(1+it, y) - \log(e^{i\theta} R_y) &= \log \zeta(1+it, y) - \log \left(\prod_{p \leq y} \left(1 - \frac{e^{i\psi}}{p}\right)^{-1} \right) + O \left(\frac{1}{\log_2 y} \right) \\ &= \sum_{p \leq y} \sum_{m \geq 1} \frac{(p^{-it})^m - (e^{i\psi})^m}{p^m m} + O \left(\frac{1}{\log_2 y} \right) \\ &= \sum_{p \leq y} \sum_{m \geq 1} \frac{(p^{-it})^m - e^{i\psi}}{p^m m} + O \left(\psi \sum_{p \leq y} \sum_{m \geq 2} \frac{1}{p^m} + \frac{1}{\log_2 y} \right) \\ (5.3) \quad &= \sum_{p \leq y} \sum_{m \geq 1} \frac{(p^{-it})^m - e^{i\psi}}{p^m m} + O \left(\frac{1}{\log_2 y} \right). \end{aligned}$$

Finally the result follows upon taking absolute values of both (5.2) and (5.3).

Now we are ready to compute the moments (1.4). Indeed we prove

Theorem 5.2. *Let $T > 0$ be large, $y \leq \log T$ a real number, and f a completely multiplicative function with values on the unit circle \mathbb{U} . If $k \leq y/(\log y)^2$ is a positive integer and $\alpha = y/k$, then*

$$\begin{aligned} I(k) &:= \frac{1}{T} \int_T^{2T} |L_f(1+it, y) + R_y|^{2k} dt \\ &= (2R_y)^{2k} \exp \left(\frac{k}{\log k} \left(-\log(\alpha) + O \left(1 + \frac{(\log \alpha)^2}{\log k} \right) \right) \right), \end{aligned}$$

Proof. First

$$\begin{aligned} I(k) &= \frac{1}{T} \int_T^{2T} (L_f(1+it, y) + R_y)^k (\overline{L_f(1+it, y)} + R_y)^k dt \\ &= \sum_{0 \leq l, m \leq k} \binom{k}{l} \binom{k}{m} R_y^{2k-l-m} \frac{1}{T} \int_T^{2T} L_f(1+it, y)^l \overline{L_f(1+it, y)}^m dt. \end{aligned}$$

Therefore by Theorem 4.1 we have

$$I(k) = \sum_{0 \leq l, m \leq k} \binom{k}{l} \binom{k}{m} R_y^{2k-l-m} \sum_{n \in S(y)} \frac{d_l(n) d_m(n)}{n^2} + O \left((R_y + 1)^{2k} \exp \left(-\frac{\log T}{4 \log_2 T} \right) \right).$$

Now $k \leq y/(\log y)^2 \leq \log T/(\log_2 T)^2$, and we know that $R_y \sim e^\gamma \log y$, thus

$$(R_y + 1)^{2k} \exp \left(-\frac{\log T}{4 \log_2 T} \right) \leq \exp \left(-\frac{\log T}{5 \log_2 T} \right).$$

We divide the main term into two parts: central terms which correspond to $l = m$, and non-central terms $l \neq m$.

The lower bound. Since the contribution of the non-central terms is positive, we have

$$I(k) \geq \sum_{0 \leq l \leq k} \binom{k}{l}^2 R_y^{2k-2l} \sum_{n \in S(y)} \frac{d_l(n)^2}{n^2} + O \left(\exp \left(-\frac{\log T}{5 \log_2 T} \right) \right).$$

Since all the terms are positive, we consider only the contribution of $l = [k/2]$. Then what remains only is to evaluate $\sum_{n \in S(y)} d_l(n)^2/n^2$. This has been done in [10] (Theorem 3). Indeed Granville and Soundararajan proved that

$$(5.4) \quad \sum_{n \in S(y)} \frac{d_l(n)^2}{n^2} = \prod_{p \leq l} \left(1 - \frac{1}{p} \right)^{-2l} \exp \left(\frac{2l}{\log l} \left(C + O \left(\frac{l}{y} + \frac{1}{\log l} \right) \right) \right),$$

where C is the same constant as (3). By (5.4) we get

$$I(k) \geq \binom{k}{l}^2 R_y^{2k-2l} \prod_{p \leq l} \left(1 - \frac{1}{p}\right)^{-2l} \exp\left(\frac{2l}{\log l} \left(C + O\left(\frac{l}{y} + \frac{1}{\log l}\right)\right)\right).$$

First by Stirling's formula we have $\binom{k}{l} \gg 2^k/\sqrt{k}$. Moreover since $\log l/\log y = 1 - \log(2\alpha)/\log y + O(1/\log^2 y)$, then

$$\prod_{p \leq l} \left(1 - \frac{1}{p}\right)^{-2l} = (R_y)^{2l} \exp\left(-2l \frac{\log(2\alpha)}{\log y} \left(1 + O\left(\frac{\log \alpha}{\log y}\right)\right)\right).$$

Thus we deduce that

$$I(k) \geq \frac{1}{\sqrt{k}} (2R_y)^{2k} \exp\left(\frac{k}{\log k} \left(-\log(\alpha) + O\left(1 + \frac{(\log \alpha)^2}{\log k}\right)\right)\right),$$

which proves the lower bound.

The upper bound. Using Cauchy's inequality, we get that

$$\begin{aligned} & \sum_{0 \leq l, m \leq k} \binom{k}{l} \binom{k}{m} R_y^{2k-l-m} \sum_{n \in S(y)} \frac{d_l(n) d_m(n)}{n^2} \\ & \leq \sqrt{\left(\sum_{0 \leq m \leq k} \sum_{0 \leq l \leq k} \binom{k}{l}^2 R_y^{2k-2l} \sum_{n \in S(y)} \frac{d_l(n)^2}{n^2} \right)^2} \\ & = (k+1) \sum_{0 \leq l \leq k} \binom{k}{l}^2 R_y^{2k-2l} \sum_{n \in S(y)} \frac{d_l(n)^2}{n^2}. \end{aligned}$$

Therefore

$$I(k) \leq (k+2) \sum_{0 \leq l \leq k} \binom{k}{l}^2 R_y^{2k-2l} \sum_{n \in S(y)} \frac{d_l(n)^2}{n^2}.$$

Thus, by (5.4) we deduce that

$$\begin{aligned} I(k) & \leq (k+2) \sum_{0 \leq l \leq k} \binom{k}{l}^2 R_y^{2k-2l} \prod_{p \leq l} \left(1 - \frac{1}{p}\right)^{-2l} \exp\left(\frac{2l}{\log l} \left(C + O\left(\frac{l}{y} + \frac{1}{\log l}\right)\right)\right) \\ & \leq R_y^{2k} \sum_{0 \leq l \leq k} \binom{k}{l}^2 \prod_{k \leq p \leq y} \left(1 - \frac{1}{p}\right)^{2l} \exp\left(O\left(\frac{k}{\log k}\right)\right) \\ & = R_y^{2k} \sum_{0 \leq l \leq k} \binom{k}{l}^2 \exp\left(\frac{2l}{\log k} \left(-\log \alpha + O\left(\frac{(\log \alpha)^2}{\log k}\right)\right) + O\left(\frac{k}{\log k}\right)\right). \end{aligned}$$

Let

$$f(l) = \binom{k}{l}^2 \exp\left(-\frac{2l \log \alpha}{\log k}\right).$$

By Stirling's formula, one has

$$\begin{aligned} \log f(l) = & 2(k \log k - (k-l) \log(k-l) - l \log l) - \log 2\pi + \log k - \log(k-l) - \log l \\ & - \frac{2l \log \alpha}{\log k} + o(1). \end{aligned}$$

Differentiating the main term of this formula with respect to l , we deduce that the maximum of f , occurs for $l = k/2(1 + O(\log \alpha / \log k))$. Thus

$$f(l) \leq 2^{2k} \exp\left(\frac{k}{\log k} \left(-\log(\alpha) + O\left(1 + \frac{(\log \alpha)^2}{\log k}\right)\right)\right),$$

which implies the upper bound.

Proof of Theorem 1. First, by Lemma 2.4 (with $A(y) = \log_2 y$) and Lemma 5.1, we have

$$\begin{aligned} M(\theta, y) = & \text{meas}\{t \in [T, 2T] : L_\psi(1 + it, y) = R_y \left(1 + O\left(\frac{1}{\log_2 y}\right)\right)\} \\ & + O\left(T \exp\left(-\frac{y \log_2 y}{10^4 \log y}\right)\right), \end{aligned}$$

where $\psi = \theta / \log_2 y$ as in Lemma 5.1. Let z be a complex number verifying $|z| \leq 1$ and $|z + 1| \geq 2 - A\epsilon$, for some positive constant A . Then $z = 1 + O(\sqrt{\epsilon})$ (where the constant in the O depends only on A). Moreover if $z - 1 = O(\epsilon)$ then $|z + 1| \geq 2 - B\epsilon$ where B depends only on the constant in the O . Thus there exist some positive constants c_1 and c_2 for which

$$(5.5) \quad M_2 + O\left(T \exp\left(-\frac{y \log_2 y}{10^4 \log y}\right)\right) \leq M(\theta, y) \leq M_1 + O\left(T \exp\left(-\frac{y \log_2 y}{10^4 \log y}\right)\right),$$

where

$$M_2 := \text{meas}\{t \in [T, 2T] : |L_\psi(1 + it, y) + R_y| \geq 2R_y \left(1 - \frac{c_2}{(\log_2 y)^2}\right)\},$$

and

$$M_1 := \text{meas}\{t \in [T, 2T] : |L_\psi(1 + it, y) + R_y| \geq 2R_y \left(1 - \frac{c_1}{4 \log_2 y}\right)\}.$$

The lower bound. For a positive integer k , we have

$$TI(k) \leq (2R_y)^{2k} M_2 + (2R_y)^{2k} \exp\left(-2c_2 \frac{k}{(\log_2 y)^2}\right) (T - M_2).$$

Now by Theorem 5.2, if $k \leq y/(\log y)^2$, and $\alpha = y/k$, we get

$$T \exp\left(\frac{k}{\log k} \left(-\log(\alpha) + O\left(1 + \frac{(\log \alpha)^2}{\log k}\right)\right)\right) - T \exp\left(-2c_2 \frac{k}{(\log_2 y)^2}\right) \leq M_2.$$

Choosing $k = [\exp(\log y - c_2 \log y/(\log_2 y)^2)]$, we deduce

$$(5.6) \quad M_2 \geq 2T \exp\left(-\exp\left(\log y - c_2 \frac{\log y}{(\log_2 y)^2}\right)\right).$$

The upper bound. Similarly for a positive integer k we have

$$(2R_y)^{2k} \exp\left(-c_1 \frac{k}{\log_2 y}\right) M_1 \leq TI(k).$$

Then if $k \leq y/(\log y)^2$, and $\alpha = y/k$, we get by Theorem 5.2 that

$$M_1 \leq T \exp\left(c_1 \frac{k}{\log_2 y} + \frac{k}{\log k} \left(-\log(\alpha) + O\left(1 + \frac{(\log \alpha)^2}{\log k}\right)\right)\right).$$

Now by choosing $k = [\exp(\log y - 2c_1 \log y/\log_2 y)]$, we have that

$$(5.7) \quad M_1 \leq \frac{1}{2} T \exp\left(-\exp\left(\log y - c_1 \frac{\log y}{\log_2 y}\right)\right).$$

Finally from (5.5), (5.6) and (5.7) we deduce that

$$T \exp\left(-\exp\left(\log y - c_2 \frac{\log y}{(\log_2 y)^2}\right)\right) \leq M(\theta, y) \leq T \exp\left(-\exp\left(\log y - c_1 \frac{\log y}{\log_2 y}\right)\right).$$

6. RANDOM EULER PRODUCTS AND THEIR DISTRIBUTION

We define $L(1, X, y) := \prod_{p \leq y} (1 - X(p)/p)^{-1}$. By the Central Limit Theorem, $L(1, X, y)$ converges to $L(1, X)$ with probability 1, as $y \rightarrow \infty$. However we want a more accurate result which quantify the rate of this convergence. Let Ω be the probability space on which $\{X(p)\}_{p \text{ prime}}$ are defined. For a real number $y > 2$, define

$$D(y) := \left\{ \omega \in \Omega : L(1, X(\omega)) = L(1, X(\omega), y) \left(1 + O\left(\frac{1}{\log y}\right)\right) \right\}.$$

Then we prove

Lemma 6.1. *Let y be large. We have*

$$1 - \text{Prob}(D(y)) \ll \exp\left(-\frac{y}{e \log y}\right).$$

Proof. First we have

$$\prod_{p>y} \left(1 - \frac{X(p)}{p}\right)^{-1} = \exp\left(\sum_{\substack{p>y \\ n \geq 1}} \frac{X(p)^n}{np^n}\right) = \exp\left(\sum_{p>y} \frac{X(p)}{p} + O\left(\frac{1}{y}\right)\right).$$

Moreover

$$\mathbb{E}\left(\left|\sum_{p>y} \frac{X(p)}{p}\right|^{2k}\right) = \mathbb{E}\left(\sum_{\substack{p_1, \dots, p_k, q_1, \dots, q_k \\ p_i, q_j > y}} \frac{X(p_1) \dots X(p_k) \overline{X(q_1) \dots X(q_k)}}{p_1 \dots p_k q_1 \dots q_k}\right).$$

Now if $p_1 \dots p_k = q_1 \dots q_k$ then $\mathbb{E}(X(p_1) \dots X(p_k) \overline{X(q_1) \dots X(q_k)}) = 1$, otherwise this expectation is 0. This gives

$$\mathbb{E}\left(\left|\sum_{p>y} \frac{X(p)}{p}\right|^{2k}\right) \ll k! \left(\sum_{p>y} \frac{1}{p^2}\right)^k \leq \left(\frac{k}{y \log y}\right)^k.$$

Thus

$$\left(\frac{1}{\log y}\right)^{2k} \text{Prob}\left(\left|\sum_{p>y} \frac{X(p)}{p}\right| > \frac{1}{\log y}\right) \leq \mathbb{E}\left(\left|\sum_{p>y} \frac{X(p)}{p}\right|^{2k}\right) \ll \left(\frac{k}{y \log y}\right)^k.$$

Finally we choose $k = y/(e \log y)$, which implies the result.

To prove Theorem 3, we have to understand the correlation between the norm and the argument of short Euler products of degree 1. For $y > 2$ define

$$P_y := \log \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1}.$$

We have

Lemma 6.2. *Let $\theta \ll 1$ and $\{x(p)\}_{p \leq y}$ a sequence of complex numbers on the unit circle, such that $\arg \prod_{p \leq y} (1 - x(p)/p)^{-1} = \theta$. If*

$$\left|\prod_{p \leq y} \left(1 - \frac{x(p)}{p}\right)^{-1}\right| = \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1} \exp(L),$$

then

$$L \leq -\frac{\theta^2}{2P_y}.$$

Moreover if y is large, there exists some real ψ verifying $\psi = \theta/P_y + O(\theta/P_y^2)$, and such that

$$\prod_{p \leq y} \left(1 - \frac{e^{i\psi}}{p}\right)^{-1} = \exp(L + i\theta) \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1},$$

with

$$L = -\frac{\theta^2}{2P_y} + O\left(\frac{\theta^2}{P_y^2}\right).$$

Proof. The first statement of the Lemma follows upon noting that $\log \prod_{p \leq y} \left(1 - \frac{x(p)}{p}\right)^{-1} = P_y + L + i\theta$, and

$$\left| \log \prod_{p \leq y} \left(1 - \frac{x(p)}{p}\right)^{-1} - \sum_{\substack{p \leq y \\ k \geq 1}} \frac{x(p)^k}{kp^k} \right| \leq P_y.$$

For the second statement, we search for ψ such that

$$\theta = \sum_{\substack{p \leq y \\ k \geq 1}} \frac{\sin(k\psi)}{kp^k}.$$

By the uniform convergence of the last series, ψ exists and we have

$$\theta = P_y \sin \psi + \sum_{\substack{p \leq y \\ k \geq 2}} \frac{\sin(k\psi) - \sin \psi}{kp^k} = \psi P_y + O(\psi + \psi^3 P_y).$$

Thus $\psi = \theta/P_y + O(\theta/P_y^2)$, and finally

$$L = \sum_{\substack{p \leq y \\ k \geq 1}} \frac{\cos(k\psi) - 1}{kp^k} = (\cos \psi - 1)P_y + \sum_{\substack{p \leq y \\ k \geq 2}} \frac{\cos(k\psi) - \cos \psi}{kp^k} = -\frac{\psi^2}{2}P_y + O(\psi^2),$$

which completes the proof.

Proof of Theorem 3. For $c > 0$, and y large enough we define the following sets

$$B_+(c, \tau, y, \theta) = \left\{ X \in \Omega : |L(1, X, y)| > e^\gamma \tau \left(1 + \frac{c}{\log y}\right) \text{ and } |\arg L(1, X, y)| > \theta + \frac{c}{\log y} \right\},$$

$$B_-(c, \tau, y, \theta) = \left\{ X \in \Omega : |L(1, X, y)| > e^\gamma \tau \left(1 - \frac{c}{\log y}\right) \text{ and } |\arg L(1, X, y)| > \theta - \frac{c}{\log y} \right\}.$$

The upper bound. If c is a sufficiently large constant, we get

$$(6.1) \quad \Phi(\tau, \theta) \leq \text{Prob}(B_-(c, \tau, y, \theta) \cap D(y)) + \text{Prob}(D^c(y)).$$

Let $C_3 > 0$ be a suitably large constant and choose

$$\tau = \log y \exp \left(-\frac{\theta^2}{2P_y} + 2C_3 \frac{\theta^2}{P_y^2} \right).$$

Take $X \in B_-(C_3, \tau, y, \theta)$, and put $L(1, X, y) = \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1} \exp(L + i\phi)$. Then $|\phi| > \theta - \frac{C_3}{\log y}$, and

$$|L(1, X, y)| > e^{\gamma\tau} \left(1 - \frac{C_3}{\log y}\right) > \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1} \exp \left(-\frac{\theta^2}{2P_y} + C_3 \frac{\theta^2}{P_y^2} \right).$$

Therefore

$$L > -\frac{\theta^2}{2P_y} + C_3 \frac{\theta^2}{P_y^2} \geq \frac{-\phi^2}{2P_y} + C_3 \frac{\theta^2}{P_y^2} - C_3 \frac{\theta}{P_y \log y} > \frac{-\phi^2}{2P_y}.$$

This contradicts Lemma 6.2, which implies that $B_-(C_3, \tau, y, \theta) = \emptyset$. Thus from (6.1) and Lemma 6.1 we deduce that

$$\Phi(\tau, \theta) \leq \text{Prob}(D^c(y)) \ll \exp \left(-\frac{y}{e \log y} \right).$$

And finally replacing y by τ , we get

$$\Phi(\tau, \theta) \leq \exp \left(-\frac{e^{\tau + \frac{\theta^2 \tau}{2 \log \tau} - 3C_3 \frac{\theta^2 \tau}{\log^2 \tau}}}{\tau} \right),$$

as desired.

The lower bound . By Lemma 6.1, if c is a sufficiently large constant, then

$$(6.2) \quad \begin{aligned} \Phi(\tau, \theta) &\geq \text{Prob}(B_+(c, \tau, y, \theta) \cap D(y)) \geq \text{Prob}(B_+(c, \tau, y, \theta)) + \text{Prob}(D(y)) - 1 \\ &\geq \text{Prob}(B_+(c, \tau, y, \theta)) - \exp \left(-\frac{y}{3 \log y} \right). \end{aligned}$$

Now put $X(p) = e^{i\theta_p}$, where the θ_p are independent random variables uniformly distributed on $(-\pi, \pi)$. Let $\tilde{\theta} = \theta(1 + 1/P_y)$. By Lemma 6.2, there exists ψ verifying

$$\psi = \frac{\tilde{\theta}}{P_y} + O \left(\frac{\tilde{\theta}}{P_y^2} \right),$$

and such that

$$\prod_{p \leq y} \left(1 - \frac{e^{i\psi}}{p}\right)^{-1} = \exp(L + i\tilde{\theta}) \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1},$$

where

$$L = -\frac{\tilde{\theta}^2}{2P_y} + O\left(\frac{\tilde{\theta}^2}{P_y^2}\right) = -\frac{\theta^2}{2P_y} + O\left(\frac{\theta^2}{P_y^2}\right).$$

We choose $X \in \Omega$ such that

$$(6.3) \quad \begin{cases} \psi - \frac{1}{(\log y)^{3/2}} < \theta_p < \psi + \frac{1}{(\log y)^{3/2}} & \text{for } p \leq z, \\ \text{and } \theta_p \in (-\pi, \pi), & \text{for } z < p \leq y, \end{cases}$$

where $z = \frac{y}{8 \log_2 y}$. In this case

$$\prod_{p \leq y} \left(1 - \frac{X(p)}{p}\right)^{-1} = \prod_{p \leq y} \left(1 - \frac{e^{i\psi}}{p}\right)^{-1} \exp\left(O\left(\frac{1}{(\log y)^{3/2}} \sum_{p \leq z} \frac{1}{p} + \sum_{z < p \leq y} \frac{1}{p}\right)\right).$$

And since

$$\sum_{z < p \leq y} \frac{1}{p} \sim \log\left(\frac{\log y}{\log z}\right) = O\left(\frac{\log_3 y}{\log y}\right),$$

then

$$\prod_{p \leq y} \left(1 - \frac{X(p)}{p}\right)^{-1} = \exp\left(-\frac{\theta^2}{2P_y} + i\left(\theta + \frac{\theta}{P_y}\right) + O\left(\frac{\theta^2}{P_y^2} + \frac{\log_3 y}{\log y}\right)\right) \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1}.$$

Let $C_4 > 0$ be a suitably large constant, and choose

$$\tau = \log y \exp\left(-\frac{\theta^2}{2P_y} - 2C_4 \frac{\theta^2}{P_y^2}\right).$$

In this case we have

$$\left| \prod_{p \leq y} \left(1 - \frac{X(p)}{p}\right)^{-1} \right| > e^{\gamma \tau} \left(1 + \frac{C_4}{\log y}\right),$$

and

$$\left| \arg \prod_{p \leq y} \left(1 - \frac{X(p)}{p}\right)^{-1} \right| > \theta + \frac{C_4}{\log y}.$$

Thus considering only these X which satisfy (6.3) we deduce that

$$\text{Prob}(B_+(c, \tau, y, \theta)) \geq \left(\frac{2}{2\pi(\log y)^{3/2}} \right)^{\pi(z)} \geq \exp \left(-\frac{y}{4\log y} + O \left(\frac{y}{\log y^2} \right) \right).$$

Finally by (6.2) we get

$$\Phi(\tau, \theta) \geq \exp \left(-\frac{y}{3\log y} \right) \geq \exp \left(-\frac{e^{\tau + \frac{\theta^2 \tau}{2\log \tau} + 3C_4 \frac{\theta^2 \tau}{\log^2 \tau}}}{\tau} \right).$$

Thus upon taking $c_3 = 3C_3$, and $c_4 = 3C_4$, we deduce the result.

Proof of Theorem 5. For the upper bound, the proof is the same as for Theorem 3, replacing Lemma 6.1 by Lemma 2.4 (taking $A(y) = \log y$). For the lower bound we use Theorems 4A and 4B to make (6.3) holds in the appropriate ranges, and follow the same lines as with Theorem 3.

7. FOURIER ANALYSIS ON THE n -DIMENSIONAL TORUS

We begin by presenting the following construction due to Barton-Montgomery-Vaaler [1]: Let $N \in \mathbb{N}$. If u, v are real numbers with $0 < u < v < 1$, we define the modified characteristic function $\phi_{u,v} : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ by

$$\phi_{u,v}(x) = \begin{cases} 1 & \text{if } u < x - n < v \text{ for } n \in \mathbb{Z}, \\ \frac{1}{2} & \text{if } u - x \in \mathbb{Z} \text{ or } v - x \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

Put $\mathbf{u} = (u_1, u_2, \dots, u_N)$ and $\mathbf{v} = (v_1, v_2, \dots, v_N)$ where $0 < u_n < v_n < 1$. If $\mathbf{L} = (L_1, \dots, L_N) \in \mathbb{N}^N$, we let $\mathcal{B}(\mathbf{L})$ to be the set of all functions $\Phi_{\mathbf{u}, \mathbf{v}} : (\mathbb{R}/\mathbb{Z})^N \rightarrow \mathbb{R}$ of the form

$$\Phi_{\mathbf{u}, \mathbf{v}}(\mathbf{x}) = \prod_{n=1}^N \phi_{u_n, v_n}(x_n),$$

and such that $(v_n - u_n)(L_n + 1) \in \mathbb{N}$ for all $1 \leq n \leq N$. The principal result of Barton-Montgomery-Vaaler is the following:

Theorem 7.1. *Let $\mathbf{L} = (L_1, \dots, L_N) \in \mathbb{N}^N$ and $\Phi_{\mathbf{u}, \mathbf{v}} \in \mathcal{B}(\mathbf{L})$.*

There exist trigonometric polynomials $A(\mathbf{x})$, $B(\mathbf{x})$ and $C(\mathbf{x})$ of N variables, with Fourier coefficients supported on the lattice

$$\mathcal{L} := \mathcal{L}(L) = \{l \in \mathbb{Z}^N : |l_n| \leq L_n, \ n = 1, 2, \dots, N\},$$

such that

$$\begin{aligned}\hat{C}(0) &= \prod_{n=1}^N \left(1 + \frac{1}{(v_n - u_n)(L_n + 1)}\right) \prod_{n=1}^N (v_n - u_n), \\ \hat{A}(0) &= \prod_{n=1}^N (v_n - u_n), \\ \hat{B}(0) &= \left(\sum_{n=1}^N \frac{1}{(v_n - u_n)(L_n + 1)}\right) \prod_{n=1}^N (v_n - u_n),\end{aligned}$$

and

$$A(\mathbf{x}) - B(\mathbf{x}) \leq \Phi_{\mathbf{u}, \mathbf{v}}(\mathbf{x}) \leq C(\mathbf{x}) \quad \text{for all } \mathbf{x} \in (\mathbb{R}/\mathbb{Z})^N.$$

Our goal is to prove Theorem 4A in the best possible uniform region for $N = \pi(y)$. To this end we prove the following Lemma which establishes the optimal choice of the lattice \mathcal{L} and thus of the degrees of the trigonometric polynomials we use later in the proof of Theorem 4A.

Lemma 7.2. *If $N = o\left(\sqrt{\log T / \log_2 T}\right)$, as $T \rightarrow \infty$, and $\{\delta_n\}_{1 \leq n \leq N}$ are real numbers between 0 and 1 such that*

$$\min_{1 \leq n \leq N} \delta_n > \delta := \left(\frac{N}{\sqrt{\log T / \log_2 T}}\right)^{2/3},$$

then there exist positive integers L_1, L_2, \dots, L_N verifying

$$(7.1) \quad p_1^{L_1} p_2^{L_2} \dots p_N^{L_N} \leq T^{1/2},$$

and

$$(7.2) \quad \sum_{n=1}^N \frac{1}{\delta_n(L_n + 1)} = o(1).$$

Moreover if (7.1) and (7.2) hold for some positive integers L_1, L_2, \dots, L_N , and any real numbers $\{\delta_n\}_{1 \leq n \leq N}$ between 0 and 1, then $N = o\left(\sqrt{\log T / \log_2 T}\right)$.

Proof. Let $L = \lfloor \log T / 2N \rfloor$. If $L_i = \lfloor L / \log p_i \rfloor$, then

$$\sum_{n=1}^N L_n \log p_n \leq LN \leq \log T / 2,$$

which implies (7.1). Moreover

$$\sum_{n=1}^N \frac{1}{\delta_n(L_n + 1)} \ll \frac{1}{\delta L} \sum_{n=1}^N \log p_n \ll \frac{N \log N}{\delta L} \ll \frac{N^2 \log N}{\delta \log T} \ll \frac{N^{4/3} \log N}{(\log T)^{2/3} (\log_2 T)^{1/3}} = o(1),$$

and so (7.2) holds.

Now suppose that there exist positive integers L_1, L_2, \dots, L_N , and real numbers $\{\delta_n\}_{1 \leq n \leq N}$ between 0 and 1, which verify (7.1) and (7.2). Then

$$\sum_{n=1}^N \frac{1}{L_n} = o(1), \quad \text{and} \quad \sum_{n=1}^N L_n \log p_n \leq \frac{\log T}{2}.$$

Thus by Cauchy's inequality, we have

$$\left(\sum_{n=1}^N \sqrt{\log p_n} \right)^2 \leq \left(\sum_{n=1}^N \frac{1}{L_n} \right) \left(\sum_{n=1}^N L_n \log p_n \right) = o(\log T).$$

Finally by partial summation we get

$$N \sqrt{\log N} \ll \sum_{n=1}^N \sqrt{\log p_n} = o(\sqrt{\log T}),$$

which implies the result.

To prove Theorem 4B we need the following Lemma

Lemma 7.3. *Assume Conjecture 2. Let $N \leq \log T / (10 \log_2 T)$, as $T \rightarrow \infty$. Put $L = [N(\log T)^2]$. If $|l_i| \leq L$, where $\{l_i\}_{1 \leq i \leq N}$ are integers not all zero, then*

$$|l_1 \log p_1 + l_2 \log p_2 + \dots + l_N \log p_N| \geq T^{-1/2}.$$

Proof. Let $\epsilon = 1/100$. Since $\{\log p\}_{p \text{ prime}}$ are linearly independent over \mathbb{Q} , there exists a constant $c > 0$ such that

$$\begin{aligned} |l_1 \log p_1 + l_2 \log p_2 + \dots + l_N \log p_N| &> \frac{c^N L}{(L^N p_1 \dots p_N)^{1+\epsilon}} \\ &> \exp(-(1+2\epsilon)N \log N - (1+\epsilon)N \log L) \\ &> \exp\left(-\frac{\log T}{2}\right). \end{aligned}$$

Proof of Theorem 4A. Let $a_j < b_j$ be the endpoints of I_j , and L_j be positive integers satisfying the conditions of Lemma 7.2. There exist integers $0 \leq r_i, s_i \leq L_i + 1$ such that $u_j := r_j / (L_j + 1) \leq a_j \leq x_j := (r_j + 1) / (L_j + 1)$ and $y_j := s_j / (L_j + 1) \leq b_j \leq v_j := (s_j + 1) / (L_j + 1)$. Thus for all $(z_1, z_2, \dots, z_N) \in (\mathbb{R}/\mathbb{Z})^N$, we have

$$\Phi_{\mathbf{x}, \mathbf{y}}(\mathbf{z}) := \prod_{j=1}^N \phi_{x_j, y_j}(z_j) \leq \prod_{j=1}^N \phi_{a_j, b_j}(z_j) \leq \Phi_{\mathbf{u}, \mathbf{v}}(\mathbf{z}) := \prod_{j=1}^N \phi_{u_j, v_j}(z_j).$$

Moreover $\Phi_{\mathbf{x}, \mathbf{y}}, \Phi_{\mathbf{u}, \mathbf{v}} \in \mathcal{B}(\mathbf{L})$. Hence

$$\int_T^{2T} \prod_{j=1}^N \phi_{x_j, y_j} \left(\left\{ \frac{t \log p_j}{2\pi} \right\} \right) dt \leq M \leq \int_T^{2T} \prod_{j=1}^N \phi_{u_j, v_j} \left(\left\{ \frac{t \log p_j}{2\pi} \right\} \right) dt.$$

Let $C(\mathbf{z})$ be the trigonometric polynomial as in Theorem 7.1, which corresponds to $\Phi_{\mathbf{u}, \mathbf{v}}$. Thus

$$\begin{aligned} M &\leq \int_T^{2T} C \left(\left\{ \frac{t \log p_1}{2\pi} \right\}, \left\{ \frac{t \log p_2}{2\pi} \right\}, \dots, \left\{ \frac{t \log p_N}{2\pi} \right\} \right) dt \\ &= \int_T^{2T} \sum_{l \in \mathcal{L}} \hat{C}(l) \exp(it(l_1 \log p_1 + \dots + l_N \log p_N)) dt \\ &= \sum_{l \in \mathcal{L}} \hat{C}(l) \int_T^{2T} \exp(it \log(p_1^{l_1} \dots p_N^{l_N})) dt. \end{aligned}$$

The diagonal term which corresponds to $l = 0$, equals $T\hat{C}(0)$. Since L_1, \dots, L_N verify the assertion (7.1) of Lemma 7.2, it follows that the off-diagonal terms contribute at most

$$\sum_{0 \neq l \in \mathcal{L}} |\hat{C}(l)| \frac{2}{\left| \log(p_1^{l_1} \dots p_N^{l_N}) \right|} \leq \left(\prod_{n=1}^N 3L_n \right) \hat{C}(0) p_1^{L_1} \dots p_N^{L_N} \leq \hat{C}(0) (p_1^{L_1} \dots p_N^{L_N})^{3/2} \leq T^{3/4} \hat{C}(0).$$

Finally since the assertion (7.2) holds for our choices of δ_j , we have

$$\begin{aligned} M &\leq T\hat{C}(0) \left(1 + O\left(T^{-\frac{1}{4}}\right) \right) \\ &= T \prod_{n=1}^N (v_n - u_n) \prod_{n=1}^N \left(1 + \frac{1}{(v_n - u_n)(L_n + 1)} \right) \left(1 + O\left(T^{-\frac{1}{4}}\right) \right) \\ &= T \left(\prod_{n=1}^N \delta_n \right) \exp \left(O \left(\sum_{n=1}^N \frac{1}{L_n} + \sum_{n=1}^N \frac{1}{\delta_n L_n} \right) \right) \left(1 + O\left(T^{-\frac{1}{4}}\right) \right) \\ &= T \left(\prod_{n=1}^N \delta_n \right) (1 + o(1)). \end{aligned}$$

For the lower bound, we follow the same lines using the corresponding trigonometric polynomials $A(\mathbf{z})$ and $B(\mathbf{z})$ for $\Phi_{\mathbf{x}, \mathbf{y}}$, as in Theorem 7.1. Indeed we have

$$\begin{aligned} M &\geq \int_T^{2T} (A - B) \left(\left\{ \frac{t \log p_1}{2\pi} \right\}, \left\{ \frac{t \log p_2}{2\pi} \right\}, \dots, \left\{ \frac{t \log p_N}{2\pi} \right\} \right) dt \\ &= T(\hat{A}(0) - \hat{B}(0)) \left(1 + O\left(T^{-\frac{1}{4}}\right) \right) \\ &= T \prod_{n=1}^N (y_n - x_n) \left(1 - \sum_{n=1}^N \frac{1}{(y_n - x_n)(L_n + 1)} \right) \left(1 + O\left(T^{-\frac{1}{4}}\right) \right) \\ &= T \left(\prod_{n=1}^N \delta_n \right) (1 + o(1)). \end{aligned}$$

This completes the proof.

Proof of Theorem 4B. The proof is exactly the same as Theorem 4A, taking $L_j = [N(\log T)^2]$ and using Lemma 7.3 instead of Lemma 7.2.

8. THE NORMAL DISTRIBUTION OF $\arg \zeta(1 + it)$

First we prove the following Lemma which shows that the dominant contribution to the $2k$ -th moment of $|\zeta(1 + it)|$ comes from the values of t for which $|\zeta(1 + it)| \approx e^\gamma \tau$, provided that $k = e^{\tau-1-C}$, where C is defined by (3).

Lemma 8.1. *Let T, τ, ϵ, k , and $\Omega_T(\tau)$ be as in Theorem 6. We have*

$$\frac{1}{T} \int_T^{2T} |\zeta(1 + it)|^{2k} dt = \frac{1}{T} \int_{\Omega_T(\tau)} |\zeta(1 + it)|^{2k} dt \left(1 + O \left(\exp \left(-\frac{2k}{(\log k)^{3/2}} \right) \right) \right).$$

Proof. Upon integrating by parts, we get

$$\begin{aligned} \frac{1}{T} \int_{\{t \in [T, 2T]: |\zeta(1+it)| < e^\gamma(\tau-\epsilon)\}} |\zeta(1 + it)|^{2k} dt &= -e^{2k\gamma} \int_0^{\tau-\epsilon} x^{2k} d\Phi_T(x) \\ (8.1) \qquad \qquad \qquad &= e^{2k\gamma} \left(-(\tau - \epsilon)^{2k} \Phi_T(\tau - \epsilon) + 2k \int_0^{\tau-\epsilon} \Phi_T(x) x^{2k-1} dx \right). \end{aligned}$$

Similarly one has

$$\begin{aligned} \frac{1}{T} \int_{\{t \in [T, 2T]: |\zeta(1+it)| > e^\gamma(\tau+\epsilon)\}} |\zeta(1 + it)|^{2k} dt \\ (8.2) \qquad \qquad \qquad &= e^{2k\gamma} \left((\tau + \epsilon)^{2k} \Phi_T(\tau + \epsilon) + 2k \int_{\tau+\epsilon}^\infty \Phi_T(x) x^{2k-1} dx \right), \end{aligned}$$

and

$$(8.3) \qquad \frac{1}{T} \int_T^{2T} |\zeta(1 + it)|^{2k} dt = e^{2k\gamma} \left(2k \int_0^\infty \Phi_T(x) x^{2k-1} dx \right).$$

In [10], Granville and Soundararajan proved that

$$2k \int_0^\infty \Phi_T(x) x^{2k-1} dx = (\log k)^{2k} \exp \left(\frac{2k}{\log k} \left(C + O \left(\frac{1}{\log k} \right) \right) \right),$$

together with

$$(8.4) \qquad \int_0^{\tau-\epsilon} \Phi_T(x) x^{2k-1} dx \ll \exp \left(-\frac{2k}{(\log k)^{3/2}} \right) \int_0^\infty \Phi_T(x) x^{2k-1} dx,$$

and

$$(8.5) \quad \int_{\tau+\epsilon}^{\infty} \Phi_T(x) x^{2k-1} dx \ll \exp\left(-\frac{2k}{(\log k)^{3/2}}\right) \int_0^{\infty} \Phi_T(x) x^{2k-1} dx.$$

By (2) we deduce that

$$\begin{aligned} \frac{(\tau+\epsilon)^{2k} \Phi_T(\tau+\epsilon)}{2k \int_0^{\infty} \Phi_T(x) x^{2k-1} dx} &= \left(\frac{\tau-C-1}{\tau+\epsilon}\right)^{-2k} \exp\left(-\frac{2e^{\tau+\epsilon-C-1}}{\tau+\epsilon} - \frac{2kC}{\log k} + O\left(\frac{k}{(\log k)^{3/2}}\right)\right) \\ &= \exp\left(\frac{2k(1+C+\epsilon)}{\tau+\epsilon} - \frac{2ke^{\epsilon}}{\tau+\epsilon} - \frac{2kC}{\log k} + O\left(\frac{k}{(\log k)^{3/2}}\right)\right) \\ &= \exp\left(\frac{2k}{\tau}(1+\epsilon-e^{\epsilon}) + O\left(\frac{k}{(\log k)^{3/2}}\right)\right) \\ &\ll \exp\left(-\frac{2k}{(\log k)^{3/2}}\right). \end{aligned}$$

Similarly we get

$$\frac{(\tau-\epsilon)^{2k} \Phi_T(\tau-\epsilon)}{2k \int_0^{\infty} \Phi_T(x) x^{2k-1} dx} \ll \exp\left(-\frac{2k}{(\log k)^{3/2}}\right).$$

Finally using equations (8.1)-(8.5), we deduce the result.

Proof of Theorem 6. Let x be a fixed real number, and define

$$\Lambda'_T(k, x) := \{t \in [T, 2T] : \frac{\arg \zeta(1+it)}{\sqrt{\frac{\log_2 k}{2k}}} < x\}.$$

We consider the following distribution function

$$\nu'_{T,k}(x) := \frac{\int_{\Lambda'_T(k,x)} |\zeta(1+it)|^{2k} dt}{\int_T^{2T} |\zeta(1+it)|^{2k} dt}.$$

The characteristic function of $\nu'_{T,k}$ is

$$\psi_{T,k}(\eta) := \frac{\int_T^{2T} |\zeta(1+it)|^{2k} \exp\left(i\eta \frac{\arg \zeta(1+it)}{\sqrt{\frac{\log_2 k}{2k}}}\right) dt}{\int_T^{2T} |\zeta(1+it)|^{2k} dt}.$$

Let $\xi = \frac{\eta}{\sqrt{\frac{\log_2 k}{2k}}}$. One can see that $\exp(i \arg \zeta(1+it)) = \zeta(1+it)^{1/2}/\zeta(1-it)^{1/2}$, which implies

$$\psi_{T,k}(\eta) = \frac{\frac{1}{T} \int_T^{2T} \zeta(1+it)^{k+\xi/2} \zeta(1-it)^{k-\xi/2} dt}{\frac{1}{T} \int_T^{2T} |\zeta(1+it)|^{2k} dt}.$$

Now uniformly for $|\xi| \leq k$, we have by Theorem 2

$$\psi_{T,k}(\eta) = \frac{\sum_{n=1}^{\infty} \frac{d_{k+\xi/2}(n) d_{k-\xi/2}(n)}{n^2}}{\sum_{n=1}^{\infty} \frac{d_k^2(n)}{n^2}} + O\left(\exp\left(-\frac{\log T}{2 \log_2 T}\right)\right).$$

Finally by Proposition 3.2, and replacing ξ by η , we deduce that uniformly for $|\eta| \leq \sqrt{\frac{\log_2 k}{2}}$, we have

$$\psi_{T,k}(\eta) = \exp\left(-\frac{\eta^2}{2} - \frac{c_0 \eta^2}{2 \log_2 k} + O\left(\frac{\eta^2}{\sqrt{\log k}} + \frac{\eta^4}{\log_2^2 k}\right)\right) + O\left(\exp\left(-\frac{\log T}{2 \log_2 T}\right)\right).$$

Let $\nu(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy$ be the normal distribution function, and $\psi(\eta) = e^{-\eta^2/2}$ its characteristic function. Then by the Berry-Esseen Theorem (Berry [2], Esseen [9]),

$$|\nu'_{T,k}(x) - \nu(x)| \leq K \int_{-R}^R \frac{|\psi_{T,k}(\eta) - \psi(\eta)|}{\eta} d\eta + \frac{B}{R},$$

for all $R > 0$, where B and K are absolute constants. We take $R = \sqrt{\frac{\log_2 k}{2}}$, which implies that

$$|\nu'_{T,k}(x) - \nu(x)| \ll \int_{-R}^R \frac{e^{-\eta^2/2} \eta^2}{\eta \log_2 k} d\eta + \frac{1}{\sqrt{\log_2 k}} \ll \frac{1}{\sqrt{\log_2 k}}.$$

Finally by Lemma 8.1, we have

$$\frac{1}{T} \int_{\Omega_T(\tau)} |\zeta(1+it)|^{2k} dt = \frac{1}{T} \int_T^{2T} |\zeta(1+it)|^{2k} dt \left(1 + O\left(\exp\left(-\frac{2k}{(\log k)^{3/2}}\right)\right)\right),$$

and

$$\begin{aligned} \frac{1}{T} \int_{\Lambda'_T(k,x)} |\zeta(1+it)|^{2k} dt - \frac{1}{T} \int_{\Lambda_T(\tau,x)} |\zeta(1+it)|^{2k} dt \\ \ll \exp\left(-\frac{2k}{(\log k)^{3/2}}\right) \frac{1}{T} \int_T^{2T} |\zeta(1+it)|^{2k} dt \end{aligned}$$

Therefore

$$\nu_{T,\tau}(x) = \nu'_{T,k}(x) + O\left(\exp\left(-\frac{2k}{(\log k)^{3/2}}\right)\right),$$

which completes the proof.

9. ANALOGOUS RESULTS FOR $L(1, \chi)$

In this section we present the analogous results for $L(1, \chi)$. Although we expect the behavior of the sets of values of $\zeta(1+it)$ and these of $L(1, \chi)$ should be the same, one should note that there are some differences between these two sets. Indeed the first set is continuous and the moments are integrals, while the second one is discrete and the moments are sums. Also an extra difficulty in the case of $L(1, \chi)$, is the possible existence of *Landau-Siegel* zeros, corresponding to exceptional *Siegel* characters χ defined as follows

$$\chi \bmod q : \text{ there exists } s \text{ with } \operatorname{Re}(s) \geq 1 - \frac{c}{\log q(\operatorname{Im}(s) + 2)} \text{ and } L(s, \chi) = 0,$$

for some small constant $c > 0$. Let S be the set of such characters. One expects this set to be empty, but what is known unconditionally (see [5]), is that such characters are very rare. Indeed each χ must be real (thus of order 2), and between any two powers of 2 there is at most one fundamental discriminant D with $(\frac{D}{\cdot}) \in S$. Throughout this section q will denote a large prime number. In this case there is at most one exceptional character χ of conductor q .

Using similar ideas, we show the existence of large values of $L(1, \chi)$ in every direction

Theorem 9.1. *Fix $\theta \in (-\pi, \pi]$. If $1 \ll y \leq \log q / \log_2 q$ is a real number, let $N(\theta, y)$ be the number of non-principal characters $\chi \notin S$ of conductor q for which*

$$L(1, \chi) = e^{i\theta} \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^{-1} \left(1 + O\left(\frac{1}{\log_2 y}\right)\right).$$

Then there exist two positive constants c_6, c_7 (depending on the constant in the O) for which

$$\phi(q) \exp\left(-y^{1-c_6/(\log_2 y)^2}\right) \leq N(\theta, y) \leq \phi(q) \exp\left(-y^{1-c_7/\log_2 y}\right).$$

Proof. We follow exactly the proof of Theorem 1: first we prove the analogue of Theorem 4.1 to get asymptotic for moments of short Euler products $\prod_{p \leq y} (1 - f(p)\chi(p)/p)^{-1}$, where f is a completely multiplicative function with values on the unit circle. Then we prove the analogue of Lemma 5.1, replacing p^{-it} by $\chi(p)$ (the proof is the same since $|\chi(p)| = 1$). What remains is to prove the analogue of Lemma 2.4, which can be done using the zero free region and zero density estimates of $L(s, \chi)$, if $\chi \notin S$.

As mentioned in the introduction, using a different approach, Granville and Soundararajan (unpublished) proved the existence of large values (and small ones) in every direction. Indeed what they established is the following

Theorem A (Gr-S). *If z is any complex number such that*

$$\frac{\pi^2}{6e^\gamma \log_2 q} \left(1 + O\left(\frac{1}{\log_3 q}\right)\right) \leq |z| \leq e^\gamma \log_2 q \left(1 + O\left(\frac{1}{\log_3 q}\right)\right),$$

then the number of non-principal characters $\chi \notin S$ of conductor q for which

$$L(1, \chi) = z \left(1 + O \left(\frac{\log_3 q}{\log_2 q} \right) \right),$$

is at least $q^{1-1/\log_2 q}$.

Also in their unpublished draft, they proved an analogue of Theorem 2 for complex moments of $L(1, \chi)$

Theorem B (Gr-S). Fix $\epsilon > 0$ and suppose that q is a sufficiently large integer. Let H be a subgroup of the character group G for $(\mathbb{Z}/q\mathbb{Z})^*$ with $[G : H] \ll \exp(\log^{\epsilon/2} q)$. Assume that there is an integer $r \leq \log^{1-\epsilon} q$ for which $\chi^r \in H$ for all $\chi \in G$. If z_1 and z_2 are complex numbers with $|z_1|, |z_2| \leq \log q / r(\log_2 q)^3$, and ξ is any character in G then, we have uniformly

$$\frac{1}{|H_\xi|} \sum_{\chi \in H_\xi} L(1, \chi)^{z_1} L(1, \bar{\chi})^{z_2} = \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{d_{z_1}(n) d_{z_2}(n)}{n^2} + o(1),$$

where H_ξ is the set of characters in ξH of order > 1 , not belonging to S .

If we restrict our selves to the case of q prime then using a similar approach as in Theorem 2, we have

Theorem 9.2. Let q be a large prime. Then uniformly for all complex numbers z_1, z_2 in the region $|z_1|, |z_2| \leq \log q / 50(\log_2 q)^2$, we have

$$\frac{1}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi \neq 1, \chi \notin S}} L(1, \chi)^{z_1} L(1, \bar{\chi})^{z_2} = \sum_{\substack{n=1 \\ (n, q)=1}}^{\infty} \frac{d_{z_1}(n) d_{z_2}(n)}{n^2} + O \left(\exp \left(-\frac{\log q}{2 \log_2 q} \right) \right).$$

Proof. We follow the same lines as the proof of Theorem 2. First by Lemma 2.3 of [11] (analogue of Lemma 2.5 for $\zeta(1+it)$), if χ is a non-principal character (mod q) not belonging to S , then

$$L(1, \chi)^z = \sum_{n=1}^{\infty} \chi(n) \frac{d_z(n)}{n} e^{-n/Z} + O \left(\frac{1}{q} \right),$$

where $Z = \exp((\log q)^{10})$ and z is any complex number with $|z| \leq (\log q)^2$. Let $k = \max\{[|z_1|] + 1, [|z_2|] + 1\}$, we have then

$$\begin{aligned} & \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi \neq 1, \chi \notin S}} L(1, \chi)^{z_1} L(1, \bar{\chi})^{z_2} \\ (9.1) \quad &= \sum_{n, m \geq 1} \frac{d_{z_1}(n) d_{z_2}(m)}{nm} e^{-(m+n)/Z} \frac{1}{\phi(q)} \sum_{\substack{\chi \pmod{q} \\ \chi \neq 1, \chi \notin S}} \chi(n) \overline{\chi(m)} + O \left(\frac{(\log 3Z)^k}{q} \right). \end{aligned}$$

We now extend the right side of (9.1) so as to include all characters (mod q). Since q is prime, S contains at most one element, thus by (3.2) the contribution of characters of S together with the principal character is bounded by

$$\frac{2}{\phi(q)} \left(\sum_{n \geq 1} \frac{d_k(n)}{n} e^{-n/Z} \right)^2 \ll \frac{(\log 3Z)^{2k}}{q}.$$

The contribution from the diagonal terms $m = n$ is

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d_{z_1}(n)d_{z_2}(n)}{n^2} e^{-2n/Z} = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{d_{z_1}(n)d_{z_2}(n)}{n^2} + O\left(\frac{\zeta(3/2)^{k^2}}{\sqrt{Z}}\right),$$

by (4.3). Using the orthogonality relations for characters, we see that the off-diagonal terms $m \neq n$ satisfy $m \equiv n \pmod{q}$ and $(mn, q) = 1$, which imply $\max(m, n) > q$. Thus the contribution of these terms is bounded by

$$(9.2) \quad 2 \sum_{n=1}^{\infty} \frac{d_k(n)}{n} e^{-n/Z} \left(\max_{b \pmod{q}} \sum_{\substack{m > q \\ m \equiv b \pmod{q}}} \frac{d_k(m)}{m} e^{-m/Z} \right).$$

Now following the proof of Proposition 3.1 (using induction on k), we can prove that

$$(9.3) \quad \max_{b \pmod{q}} \sum_{\substack{m > q \\ m \equiv b \pmod{q}}} \frac{d_k(m)}{m} e^{-m/Z} \leq \frac{(\log 3Z)^k}{y},$$

where $y = \exp(\log q / \log_2 q)$. Finally by (9.2) and (9.3) we deduce the result.

Using Fourier analysis on the n -dimensional torus, and the construction of Barton-Montgomery-Vaaler [1], we proved the uniform distribution of the values $\{p^{it} : t \in [T, 2T]\}_{p \leq y}$. We can use exactly the same ideas to prove that the values $\{\chi(p) : \chi \pmod{q}\}_{p \leq y}$ have the same behavior. Indeed we have

Theorem 9.3. *Let $2 < y$ be a real number. For each $1 \leq j \leq \pi(y)$, let $I_j \subset (0, 1)$ be an open interval of length $\delta_j > 0$. Define*

$$N(I_1, \dots, I_{\pi(y)}) = N := \left| \left\{ \chi \pmod{q} : \left\{ \frac{\arg(\chi(p_j))}{2\pi} \right\} \in I_j, \text{ for all } 1 \leq j \leq \pi(y) \right\} \right|,$$

where p_j is the j -th smallest prime, and $\{\cdot\}$ denotes the fractional part. We have

$$N \sim \phi(q) \prod_{j \leq \pi(y)} \delta_j,$$

uniformly for $y \leq \sqrt{\log q}/(\log_2 q)^2$, and $\delta_j > (\log_2 q)^{-5/3}$.

One should note that there is no analogue of Theorem 4B (where we assume Conjecture 2) in this case.

Proof. The proof is exactly the same as Theorem 4A, noting that

$$(9.4) \quad \sum_{\chi \pmod{q}} A\left(\frac{\arg(\chi(p_1))}{2\pi}, \dots, \frac{\arg(\chi(p_n))}{2\pi}\right) = \phi(q) \hat{A}(0),$$

if A is a trigonometric polynomial in n variables, with Fourier coefficients supported in a lattice

$$\mathcal{L} = \{l \in \mathbb{Z}^n : |l_i| \leq L_i, i = 1, 2, \dots, n\},$$

with $p_1^{L_1} p_2^{L_2} \dots p_n^{L_n} \leq q$. This follows from the orthogonality relation for characters and the fact that

$$\begin{aligned} & \sum_{\chi \pmod{q}} A\left(\frac{\arg(\chi(p_1))}{2\pi}, \dots, \frac{\arg(\chi(p_n))}{2\pi}\right) \\ &= \sum_{\chi \pmod{q}} \sum_{l \in \mathcal{L}} \hat{A}(l) \exp(i(l_1 \arg(\chi(p_1)) + \dots + l_n \arg(\chi(p_n)))) = \sum_{l \in \mathcal{L}} \hat{A}(l) \sum_{\chi \pmod{q}} \chi\left(\prod_{i=1}^n p_i^{l_i}\right). \end{aligned}$$

Finally we can use the same ideas in the proofs of Theorems 5 and 6, to deduce analogous results for $L(1, \chi)$. Indeed define

$$\Phi_q(\tau) := \frac{1}{\phi(q)} |\{\chi \pmod{q}, \chi \neq 1, \chi \notin S : |L(1, \chi)| > e^\gamma \tau\}|, \text{ and}$$

$$\Phi_q(\tau, \theta) := \frac{1}{\phi(q)} |\{\chi \pmod{q}, \chi \neq 1, \chi \notin S : |L(1, \chi)| > e^\gamma \tau, |\arg L(1, \chi)| > \theta\}|.$$

In [10], Granville and Soundararajan proved that the asymptotic relation (2) holds also for $\phi_q(\tau)$. For $\Phi_q(\tau, \theta)$, similarly to Theorem 5 we prove

Theorem 9.4. *Let q be a large prime number. There exist two positive constants c_8 and c_9 such that*

$$\Phi_q(\tau, \theta) \leq \exp\left(-\frac{e^{\tau + \frac{\theta^2 \tau}{2 \log \tau} - c_8 \frac{\theta^2 \tau}{\log^2 \tau}}}{\tau}\right),$$

uniformly for $1 \ll \tau \leq \log_2 q$, and $(\log \tau) \sqrt{\frac{\log_2 \tau}{\tau}} < \theta \ll 1$. And

$$\Phi_q(\tau, \theta) \geq \exp\left(-\frac{e^{\tau + \frac{\theta^2 \tau}{2 \log \tau} + c_9 \frac{\theta^2 \tau}{\log^2 \tau}}}{\tau}\right),$$

uniformly for $1 \ll \tau \leq (\log_2 q)/2 - 2 \log_3 q$ and $(\log \tau) \sqrt{\frac{\log_2 \tau}{\tau}} < \theta \ll 1$.

Proof. For the upper bound, the proof is the same as for Theorem 5, replacing Lemma 6.1 by the analogue of Lemma 2.4 for $L(1, \chi)$ (taking $A(y) = \log y$). For the lower bound we use Theorem 9.3 to make (6.3) holds and follow the same lines as with Theorem 5.

Corollary 9.1. *If $1 \ll \tau \leq \log_2 q - \log_3 q$, then for almost all characters $\chi \pmod q$, with $|L(1, \chi)| > e^\gamma \tau$, we have $|\arg L(1, \chi)| \leq (\log \tau) \sqrt{\log_2 \tau / \tau}$.*

We prove also

Theorem 9.5. *Let q be a large prime number, $1 \ll \tau \leq \log_2 q - 3 \log_3 q$ a real number, $\epsilon = \tau^{-1/5}$ and $k = e^{\tau-1-C}$, where C is defined by (3). Let*

$$\Omega_q(\tau) := \{\chi \pmod q, \chi \neq 1, \chi \notin S : e^\gamma(\tau - \epsilon) \leq |L(1, \chi)| \leq e^\gamma(\tau + \epsilon)\},$$

and for a real number x , let

$$\Lambda_q(\tau, x) := \{\chi \in \Omega_q(\tau) : \frac{\arg L(1, \chi)}{\sqrt{\frac{\log(\tau-1-C)}{2e^{\tau-1-C}}}} < x\} \quad \text{and} \quad \nu_{q,\tau}(x) := \frac{\sum_{\chi \in \Lambda_q(\tau, x)} |L(1, \chi)|^{2k}}{\sum_{\chi \in \Omega_q(\tau)} |L(1, \chi)|^{2k}}.$$

Then we have

$$\nu_{q,\tau}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy + O_x\left(\frac{1}{\sqrt{\log \tau}}\right).$$

Proof. The proof is exactly the same as Theorem 6, using Theorem 9.2, along with Proposition 3.2 and the results of Granville-Soundararajan [10] for the distribution of $|L(1, \chi)|$ (which are exactly the same as for $|\zeta(1+it)|$).

REFERENCES

- [1] J.T. Barton, H.L. Montgomery and J.D. Vaaler, *Note on a Diophantine inequality in several variables*, Proc. Amer. Math. Soc. **129**, (2001), 337-345.
- [2] A.C. Berry, *The accuracy of the Gaussian approximation to the sum of independent variables*, Trans. Amer. Math. Soc. **49**, (1941), 122-136.
- [3] E. Bombieri, *Le grand crible en théorie analytique des nombres*, Astérisque **18**, (1987/1974), 103 pp.
- [4] J. Cogdell and P. Michel, *On the complex moments of symmetric power L -functions at $s = 1$* , IMRN **31**, (2004), 1561-1617.
- [5] H. Davenport, *Multiplicative number theory*, Springer Verlag, New York, 1980..
- [6] W. Duke, *Extreme values of Artin L -functions and class numbers*, Compositio Math. **136**, (2003), no. 1, 103-115.
- [7] P.D.T.A. Elliott, *On the size of $L(1, \chi)$* , J. reine angew. Math. **236**, (1969), 26-36.
- [8] P.D.T.A. Elliott, *On the distribution of the values of quadratic L -series in the half-plane $\sigma > 1/2$* , Invent. Math. **21**, (1973), 319-338.
- [9] C.G. Esseen, *Fourier analysis of distribution functions, A mathematical study of the Laplace-Gaussian law*, Acta Math. **77**, (1945), 1-125.
- [10] A. Granville and K. Soundararajan, *Extreme values of $|\zeta(1+it)|$* , The Riemann zeta function and related themes: papers in honour of Professor K. Ramachandra, Ramanujan Math. Soc. Lect. Notes Ser., **2**, (2006), 65-80.
- [11] A. Granville and K. Soundararajan, *The distribution of values of $L(1, \chi_d)$* , Geometric and Funct. Anal. **13**, (2003), 992-1028.
- [12] L. Habsieger and E. Royer, *L -functions of automorphic forms and combinatorics: Dyck paths*, Ann. Inst. Fourier (Grenoble) **54**, (2004), no.7, 2105-2141.

- [13] S. Lang, *Elliptic curves: Diophantine analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **231**. Springer-Verlag, Berlin-New York, 1978.
- [14] Y.-K. Lau and J. Wu, *Extreme values of symmetric power L -functions at 1*, Acta Arith. **126**, (2007), no.1, 57-76.
- [15] J.E. Littlewood, *On the function $1/\zeta(1+it)$* , Proc. London Math. Soc. **27**, (1928), 349-357.
- [16] J.E. Littlewood, *On the class number of the corpus $P(\sqrt{-k})$* , Proc. London Math. Soc. **27**, (1928), 358-372.
- [17] J.Y. Liu, E. Royer and J. Wu, *On a conjecture of Montgomery-Vaughan on extreme values of automorphic L -functions at 1. To appear at proceedings for the conference on "Anatomy of integers", Montreal, (2006).*
- [18] H.L. Montgomery, *Ten lectures in the interface between Analytic Number Theory and Harmonic Analysis*, American Mathematical Society, Providence, RI, (1994).
- [19] H.L. Montgomery and R.C. Vaughan, *Extreme values of Dirichlet L -functions at 1*, "Number Theory in Progress" (K. Györy, H. Iwaniec, J. Urbanowicz, eds.), de Gruyter, Berlin, (1999), 1039-1052.
- [20] K.K. Norton, *Upper bounds for sums of powers of divisor functions*, J. Number Theory **40**, (1992), 60-85.
- [21] E. Royer, *Statistique de la variable aléatoire $L(\text{sym}^2 f, 1)$* , Math. Ann. **321**, (2001), no. 3, 667-687.
- [22] E. Royer, *Interprétation combinatoire des moments négatifs des valeurs de fonctions L au bord de la bande critique*, Ann. Sci. École Norm. Sup. (4) **36**, (2003), no. 4, 601-620.
- [23] E. Royer and J. Wu, *Taille des valeurs de fonctions L de carrés symétriques au bord de la bande critique*, Rev. Mat. Iberoamericana **21**, (2005), no. 1, 263-312.
- [24] E. Royer and J. Wu, *Special values of symmetric power L -functions and Hecke eigenvalues*, J. Théor. Nombres Bordeaux **19**, (2007), no. 3, 703-753.
- [25] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford University Press, Oxford, 1986.

DÉPARTEMENT DE MATHÉMATIQUES ET STATISTIQUE, UNIVERSITÉ DE MONTRÉAL, CP 6128 SUCC
CENTRE-VILLE, MONTRÉAL, QC H3C 3J7, CANADA
E-mail address: Lamzouri@dms.umontreal.ca